
Mathematical Reviews

Vol. 5, No. 7

July-August, 1944

pp. 169-196

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, PRINCE and LEMON STREETS, LANCASTER, PENNSYLVANIA

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
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THE LONDON MATHEMATICAL SOCIETY
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Editorial Office

MATHEMATICAL REVIEWS, BROWN UNIVERSITY, PROVIDENCE, R. I.

Subscriptions: Price \$13 per year (\$6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, LANCASTER, PENNSYLVANIA, or BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 1, section 534, P. L. and R. authorized November 9, 1940.

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ALGEBRA

Mann, H. B. On the construction of sets of orthogonal Latin squares. *Ann. Math. Statistics* 14, 401-414 (1943). [MF 9845]

This is a clear exposition of the application of elementary number theory and group theory to the construction of generalized Eulerian squares. After remarking that the Abelian group of order $m=p^a$ and type (1^a) admits an automorphism of period $m-1$, say s , the author proves [theorem 3] that the multiplication table of the elements $1, P, P^s, P^{s^2}, \dots, P^{s^{m-2}}$ forms a Latin square (of order m) which leads, by cyclic permutation of the last $m-1$ rows, to a set of $m-1$ mutually orthogonal Latin squares. By an extension of this method to the direct product of n such groups, he constructs r mutually orthogonal Latin squares of order $p_1^{a_1} \dots p_n^{a_n}$, where $r = \min(p_i^{a_i} - 1)$, for example, 3 such squares of order 20. [There are misprints in the third and fourth sentences after table I on page 412.]

H. S. M. Coxeter (Toronto, Ont.).

Sispanov, Sergio. On the trinomic equation of fifth degree. *Bol. Mat.* 16, 153-159 (1943). (Spanish) [MF 9920]

Abramescu, N. Application of geometry to the discussion and separation of roots of an equation. *Gaz. Mat.* 48, 6 pp. (1943). (Roumanian French summary) [MF 9430]

The author illustrates by examples the following procedure of separating roots of equations. We solve the given equation $f(x)=0$ with respect to a variable parameter if there is one, or with respect to one of the coefficients, obtaining $k=P(x)$. Constructing the curve $y=P(x)$ we find its intersections with the line $y=k$.

I. J. Schoenberg.

Mohr, Ernst. Beweis des sogenannten Fundamentalsatzes der Algebra im reellen Gebiete. *J. Reine Angew. Math.* 184, 175-177 (1942). [MF 9033]

Every polynomial with real coefficients can be expressed as the product of quadratic factors with real coefficients if it is of even degree, and of such a product times a linear factor if it is of odd degree. This is proved without introducing complex variables.

A. C. Schaeffer.

Eilenberg, Samuel and Niven, Ivan. The "fundamental theorem of algebra" for quaternions. *Bull. Amer. Math. Soc.* 50, 246-248 (1944). [MF 10206]

A polynomial in quaternions is a sum of a finite number of monomials of the form $a_0 x_1 x_2 \dots x_n$, where x and a_i are real quaternions. It is here proved that, if such a polynomial has but one term of the highest degree n , then it always has at least one root. For the special case where each term is of the form ax^n this had been proved by Niven [*Amer. Math. Monthly* 48, 654-661 (1941); these *Rev.* 3, 264] by essentially algebraic methods. The proof here given for the general case is topological. It is shown that the polynomial maps into itself the four-dimensional sphere of all real quaternions, with the point infinity added; and that the degree of this map (in the sense of Brouwer) is n . This

is accomplished by showing that the given polynomial and the polynomial x^n give homotopic maps and that the map of the latter has degree n .

H. W. Brinkmann.

Fine, N. J. and Niven, Ivan. The probability that a determinant be congruent to $a \pmod{m}$. *Bull. Amer. Math. Soc.* 50, 89-93 (1944). [MF 9891]

Of the m^{n^2} m -rowed square matrices with integral elements a_{ij} , $0 \leq a_{ij} < m$, let g be the number whose determinants are congruent to a modulo m . The authors obtain formulas for determining the "probability" $P_n(a, m) = g/m^{n^2}$ in all cases.

R. Hull (Lincoln, Neb.).

Williamson, John. Hadamard's determinant theorem and the sum of four squares. *Duke Math. J.* 11, 65-81 (1944). [MF 10148]

The question of the existence of an orthogonal matrix of order n each of whose elements is ± 1 has been considered by Hadamard, Paley and others. Paley showed that such a matrix of order $n > 2$ can exist only if $4|n$, and used Legendre symbols in $GF(p^2)$ to construct such matrices for certain values of n including all multiples of 4 less than 200 except 92, 116, 156, 172, 184 and 188. In the first part of the present paper, the author improves Paley's methods and results but removes none of the exceptional values less than 200. In the second part the author uses an entirely new method involving integral quaternions and shows that a Hadamard matrix of order 172 exists. Moreover he finds several solutions for certain values of n where Paley had found but one solution.

C. C. MacDuffee.

Angheluta, Th. On orthogonal transformations whose matrices are symmetric. *Revista Mat. Timisoara* 21, 3-10 (1941). (Roumanian) [MF 9420]

In the space of an n -dimensional Cartesian coordinate system we consider the linear point to point transformation which expresses the geometric symmetry in the plane $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$. For evident a priori geometric reasons this transformation enjoys the following properties. (1) It is orthogonal. (2) Its matrix is symmetric. The author shows that this transformation, which depends on $n-1$ real parameters, is the most general linear transformation enjoying properties (1) and (2). By simply counting available parameters we seem to get a different result. Indeed, the real orthogonal matrices depend on $n(n-1)/2$ parameters. As symmetry imposes $n(n-1)/2$ conditions, we seem to be left with no free parameters. The author explains this apparent contradiction.

I. J. Schoenberg.

Abstract Algebra

Everett, C. J. Sequence completion of lattice moduls. *Duke Math. J.* 11, 109-119 (1944). [MF 10152]

The author studies the completion of a lattice modul (or lattice ordered Abelian group) L with respect to the

σ -convergence of G. Birkhoff. The Cantor extension L' by σ -regular sequences is σ -complete if L admits a diagonal process, but not in general otherwise. The MacNeille completion [Trans. Amer. Math. Soc. 42, 416-460 (1937)], restricted to cuts possessing inverses, yields an σ -complete group M containing L' , L'' , etc., and presumably $L^{(\lambda)} = M$ for some ordinal λ . *I. Kaplansky* (Cambridge, Mass.).

Wilcox, L. R. Modularity in Birkhoff lattices. Bull. Amer. Math. Soc. 50, 135-138 (1944). [MF 9899]

The author has previously introduced the concept of an M -symmetric lattice: M -symmetry means that $(a \cup b) \cap c = a \cup (b \cap c)$ for all $a \leq c$ implies $(b \cup a) \cap c = b \cup (a \cap c)$ for all $b \leq c$. The author shows that a lattice of "finite-dimensional type" is M -symmetric if and only if it is upper semi-modular (a "Birkhoff lattice") in the sense of the reviewer's "Lattice Theory" [Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1940; these Rev. 1, 325]. In the infinite-dimensional case, an M -symmetric lattice is still upper semi-modular, but there exist upper semi-modular lattices which are not M -symmetric. An example is given to support the contention that the concept of M -symmetry is more significant in the infinite-dimensional case.

G. Birkhoff (Cambridge, Mass.).

Transue, W. R. Remarks on transitivities of betweenness.

Bull. Amer. Math. Soc. 50, 108-109 (1944). [MF 9894]

V. Glivenko has defined the betweenness relation abc in a general lattice L to mean $(a \cap b) \cup (b \cap c) = b = (a \cup b) \cap (b \cup c)$. E. Pitcher and M. F. Smiley [Trans. Amer. Math. Soc. 52, 95-114 (1942); these Rev. 4, 87] have discussed the validity of ten properties T1-T10 of the betweenness relation in simply ordered sets ("chains") which involve five elements and are not equivalent to logical combinations of properties involving only four points. The author shows that T8 and T9 hold separately if and only if L is a chain; T10 holds also if L is the sum of two chains, joined at 0 and 1.

G. Birkhoff (Cambridge, Mass.).

Ore, Oystein. Some studies on closure relations. Duke Math. J. 10, 761-785 (1943). [MF 9759]

A " C -space" is a set with a closure operation satisfying $\bar{A} = \bar{A}$, $\bar{A} \supseteq A$, $\bar{0} = 0$ and $A \supseteq B$ implies $\bar{A} \supseteq \bar{B}$. Some of the elementary ideas of the usual theory of T -spaces (concepts of T_0 -space, T_1 -space, relativization) are extended to C -spaces. Necessary and sufficient conditions are found for relativization to yield an isomorphic lattice of closed sets. Various simple considerations yield numerous other results correlating the existence of irreducible elements with infinite distributivity or involving duality and bicompleteness. We cite: any relativization is composed of a relativization with respect to a closed subset and a dense relativization; any lattice can be embedded in a self-dual lattice; the lattice of all intervals of a complete lattice is "bicomplete of class two" (this means that $\wedge (a_i/b_i) = 0$ implies that, for some pair of intervals a_i/b_i and a_j/b_j , $(a_i/b_i) \cap (a_j/b_j) = 0$).

G. Birkhoff (Cambridge, Mass.).

Ore, Oystein. Combinations of closure relations. Ann. of Math. (2) 44, 514-533 (1943). [MF 8877]

The author discusses closure operations $x: A \rightarrow x(A)$ on subsets of a set S (requiring $A \subseteq x(A)$, $x(x(A)) = x(A)$, and

that $A \subseteq B$ implies $x(A) \subseteq x(B)$). It is known that, if $x \leq y$ is defined to mean $x(A) \subseteq y(A)$ for all A , the closure operations on S form a complete lattice (structure) C_S . The author shows that any element of C_S is a meet of dual points, that for every x there is a greatest y such that $x \cap y = 0$ and that C_S is upper semi-modular ("Birkhoff condition"). Given a, c , $a \cap (c \cup x) = (a \cap c) \cup x$ for every $x \subseteq a$ if and only if $c \cap (a \cup x) = (c \cap a) \cup x$ for every $x \subseteq c$. Closure relations can be multiplied under the definition $ab(A) = b(a(A))$; ab is not generally a closure relation; $ab = ba$ if and only if both are closure relations. The automorphisms of C_S are all induced by permutations of the points of S ; homomorphisms are also discussed.

G. Birkhoff (Cambridge, Mass.).

Stabler, E. R. Boolean representation theory. Amer. Math. Monthly 51, 129-132 (1944). [MF 10163]

A composite proof, using methods due to several different authors, of Stone's theorem on the representation of a Boolean ring as a ring of sets. Zorn's lemma is used to assure the existence of the required maximal ideals.

N. H. McCoy (Northampton, Mass.).

Duffin, R. J. and Pate, Robert S. An abstract theory of the Jordan-Hölder composition series. Duke Math. J. 10, 743-750 (1943). [MF 9757]

The authors have formulated in terms of lattice theory a theorem which reduces in the case of groups to the Jordan-Hölder theorem for composition series. This is accomplished through the concept of an operator lattice: An operator lattice is a Boolean algebra in which is defined a multiplication which is distributive with respect to union. An element A is left normal under an operator lattice \mathfrak{S} if it satisfies the following six axioms, where X and Y are any elements of \mathfrak{S} and 0 is the null set: (1) $AX = A$; (2) if $X \neq 0$, then $AX \neq 0$ and $XA \neq 0$; (3) $(AX) \cap Y = 0$ implies $(AY) \cap X = 0$; (4) $A(XY) = (AX)Y$; (5) $A(XY) \supseteq X(AY)$; (6) $(AX) \cap (AY) = 0$ implies $(XA) \cap (YA) = 0$. Using this definition of normality the authors prove a Jordan-Hölder theorem for an operator lattice which may be specialized to give the well-known theorem on composition series either for ordinary groups or for the various generalized groups such as quasigroups, multigroups and semigroups. Their theorem contains the previous similar results obtained in the case of multigroups by Drescher and Ore and for certain types of quasigroups by the reviewer.

D. C. Murdoch.

Knies, Guillermo. Representation of a ring connected with the spin 1. Revista Union Mat. Argentina 9, 113-117 (1943). (Spanish) [MF 9853]

It is known that three elements $\beta_1, \beta_2, \beta_3$ which satisfy

$$(1) \quad \beta_1\beta_2\beta_3 + \beta_2\beta_3\beta_1 = \beta_1\delta_{23} + \beta_2\delta_{13}$$

and the spin 1 commutation relationship

$$(2) \quad \beta_1\beta_2 - \beta_2\beta_1 = i\beta_3$$

(and cyclically) generate a ring of 10 members. The author shows that three elements which satisfy (1) but not (2) generate a ring of 35 members. This ring has representations of orders 5, 3 and 1. There are several typographical errors. In particular, the last equation on page 115 should read $\beta_1\beta_2 = \beta_2\beta_1$.

A. Schwartz (State College, Pa.).

Sagastume Berra, A. E. Representation of matrix algebras as crossed products. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie 2: Revista 2, 365-381 (1943). (Spanish) [MF 10068]

Let G , of order n , be the Galois group of the field \mathfrak{K} which is normal over the fundamental field \mathfrak{F} . Then there exist matrices S_1, \dots, S_n of order n over \mathfrak{F} which constitute a group isomorphic to G , and a field $\mathfrak{K}^* \cong \mathfrak{K}$ consisting of matrices of order n with elements in \mathfrak{F} and such that every matrix of order n with elements in \mathfrak{F} can be expressed uniquely in the form $\sum_{i=1}^n S_i h_i$, where the h_i are uniquely determined matrices of the field \mathfrak{K}^* . The automorphisms of \mathfrak{K}^* over \mathfrak{F} are given by $h_i \mapsto S_j^{-1} h_i S_j$. This is an extension to the case of an arbitrary finite group G of a known result for the case in which G is cyclic. N. H. McCoy.

Sagastume Berra, A. E. Dihedral algebras. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie 2: Revista 2, 383-393 (1943). (Spanish) [MF 10069]

A normal simple algebra \mathfrak{A} of degree n over the field \mathfrak{F} is called a crossed product if \mathfrak{A} has a subfield \mathfrak{K} of degree n over \mathfrak{F} which is normal over \mathfrak{F} . [See, for example, A. A. Albert, Structure of Algebras, Amer. Math. Soc. Colloquium Publ., v. 24, New York, 1939, chap. V; these Rev. 1, 99.] The author characterizes crossed products for the case in which the Galois group of \mathfrak{K} over \mathfrak{F} is a dihedral group and calls such algebras dihedral algebras. In particular, he gives necessary and sufficient conditions that a dihedral algebra shall be a complete matrix algebra or a division algebra. N. H. McCoy (Northampton, Mass.).

Baer, Reinhold. Radical extensions and crossed characters. Bull. Amer. Math. Soc. 49, 701-710 (1943). [MF 9307]

The author makes use of his theory of crossed characters [Trans. Amer. Math. Soc. 54, 103-170 (1943); these Rev. 5, 59] to obtain results concerning radical extensions. Let K be a normal separable extension of a field F with characteristic prime to m , and let E be the group of m th roots of unity in K . A C -character of the Galois group of K/F is a function from G to E satisfying $f(u)f(v)=f(uv)$; G is C -complete if only the identity is mapped into 1 by every C -character. The C -completeness of G is a necessary and sufficient condition for K to be obtainable from F by adjunction of m th roots. Let K now be such an extension by m th roots further containing all m th roots of unity (an " m -extension"); then there exists a group S between F^* (the multiplicative group of F) and F^{*m} such that $K=F(S^{1/m})$. The uniqueness of S is known if F contains the m th roots of unity, that is, if the Galois group over F of $x^m-1=0$ is the identity; the author shows that certain weaker assumptions concerning this Galois group suffice to establish the uniqueness of S . I. Kaplansky.

Chevalley, Claude. On the theory of local rings. Ann. of Math. (2) 44, 690-708 (1943). [MF 9406]

This material is preparatory to a forthcoming local theory of algebraic varieties and consists of the development of a number of properties of semi-local and local rings. All the rings considered are commutative, have a unit element and have the property that every ideal has a finite set of generators. A ring \mathfrak{o} is semi-local if it has only a finite set $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of maximal prime ideals. Put $\mathfrak{a}=\mathfrak{p}_1 \cdots \mathfrak{p}_n$. With \mathfrak{a}' as a sequence of neighborhoods of \mathfrak{o} a topology is defined in \mathfrak{o} . In terms of this topology and a strong defini-

tion of convergence ($\sum a_n$ converges whenever $a_n \rightarrow 0$) the ring has a unique completion in which every convergent sequence has a limit. In developing the theory of a complete semi-local ring \mathfrak{o} use is made of the homomorphisms of the ring $\mathfrak{o}[[X_1, \dots, X_r]]$ of formal power series over \mathfrak{o} into \mathfrak{o} defined by mapping $F(X_1, \dots, X_r)$ into $F(a_1, \dots, a_r)$, where the a_i are elements of \mathfrak{o} .

A local ring \mathfrak{o} is defined as a ring in which the nonunits form an ideal \mathfrak{m} and which contains a subfield K such that $\mathfrak{o}/\mathfrak{m}$ is a finite K -module. The integer r is the dimension of \mathfrak{o} if there exist r elements, but no fewer, generating an ideal primary for \mathfrak{m} . Such a set of r elements is called a system of parameters for \mathfrak{o} . The multiplicity (or degree of ramification) of \mathfrak{o} with respect to a system of parameters x_1, \dots, x_r is defined to be $[\mathfrak{o} : K[[x_1, \dots, x_r]]/[\mathfrak{o}/\mathfrak{m} : K]]$, which quantity is shown to be an integer. R. J. Walker.

Chevalley, Claude. A new kind of relationship between matrices. Amer. J. Math. 65, 521-531 (1943). [MF 9383]

If X and Y are matrices of degree m and n , respectively, the Kronecker sum $X+Y$ is the matrix $X \times E_n + E_m \times Y$, where the \times denotes Kronecker multiplication and the E 's are identity matrices [cf. Zassenhaus, Abh. Math. Sem. Hansischen Univ. 13, 1-100 (1939), in particular, p. 26]. Let $X^* = -X'$, the negative transposed, and set

$$X_{r,s} = X^* + \dots + X^* + X + \dots + X,$$

with r terms X^* and s terms X . A vector e in the m^{r+s} dimensional space is called an invariant of X if $X_{r,s}e=0$ and a matrix Y is called a replica of X if every invariant of X is also an invariant of Y . If the underlying field is perfect the problem of determining the replicas can be reduced to the two special cases where X is (1) semi-simple and (2) nilpotent. The semi-simple case is easily disposed of. If X is nilpotent and the field is of characteristic 0, any replica of X is a scalar multiple of X . It is also proved for fields of characteristic 0 that X is nilpotent if and only if $\text{trace } XY=0$ for all replicas Y of X . N. Jacobson.

Scholz, Arnold. Zur Idealtheorie in unendlichen algebraischen Zahlkörpern. J. Reine Angew. Math. 185, 113-126 (1943). [MF 9594]

Let K be an infinite extension of an algebraic number field P . The author studies the decomposition of a prime ideal \mathfrak{p} of P into its primary components in K . In particular, a connection is established between (i) the ideal theory in K as developed by Chevalley and Krull [see, for example, W. Krull, Math. Z. 29, 42-54 (1928); 31, 527-557 (1930)] and (ii) the infinite Galois theory as given by Krull and Herbrand. This way a certain uniformity is achieved. The paper contains some interesting examples illustrating the infinite Hilbert theory. O. F. G. Schilling (Chicago, Ill.).

Eaton, J. E. A Galois theory for differential fields. Duke Math. J. 10, 751-760 (1943). [MF 9758]

The author seeks to develop a Galois theory for algebraically transcendental extensions to partial differential fields. He considers a prime differential ideal of differential polynomials and investigates the manner in which the ideal decomposes when the coefficient field is extended. It is known that extensions which produce such a decomposition are characterized by the fact that they render reducible every resolvent associated with the prime ideal and are consequently algebraic extensions. J. F. Ritt.

NUMBER THEORY

Kesava Menon, P. Identities in multiplicative functions.

J. Indian Math. Soc. (N.S.) 7, 58-62 (1943). [MF 9416]

The author proves some identities among the number theoretical functions $\lambda_k(M)$, $E_k(M)$, $e(M)$, $\sigma_k(M)$ which are defined as follows. Let $M = p_1^{r_1} p_2^{r_2} \dots p_t^{r_t}$ be the factorization of M . Then $\lambda_k(M) = k^{r_1+r_2+\dots+r_t}$, $E_k(M) = k^t$, $e(M) = \prod_{i=1}^t (1/r_i!)$; $\sigma_k(M)$ denotes the number of representations of M as product of k factors (including unity); representations which differ only by the order of the factors are considered as different. A. Brauer (Chapel Hill, N. C.).

Roussel, André. Sur certaines applications arithmétiques de la théorie des résidus. C. R. Acad. Sci. Paris 216, 20-21 (1943). [MF 9995]

In order to find the number of times a function of a complex variable $f(z)$, holomorphic in an area A bounded by a contour C , takes on integral values in A , the author observes that this number k is equal to the number of real roots interior to A of $F(z) = \sin \pi f(z)$. Consequently this number may be expressed by a curvilinear integral following a classical theorem. This method is applied to the problem of finding the number k of divisors of a given integer p . It leads to the equation

$$k = \frac{1}{2\pi i} \int_C z^{-2} \cot(\pi f(z)) dz,$$

where the closed contour C is suitably chosen so as to enclose small intervals of the real axis about the positive integers from 2 to N ($N > p$) but enclosing no other portion of the real axis. Other simple applications are discussed.

M. S. Robertson (New Brunswick, N. J.).

Erdős, P. Corrections to two of my papers. Ann. of Math. (2) 44, 647-651 (1943). [MF 9403]

[For the review of the first correction, cf. these Rev. 5, 180.] The second correction refers to the paper "On some asymptotic formulas in the theory of factorisation numerorum" [Ann. of Math. (2) 42, 989-993 (1941); these Rev. 3, 165], in which the main theorem was stated incorrectly. It should read as follows. Let $1 < a_1 < a_2 < \dots$ be a sequence of integers such that, for some ρ , $\sum a_k^{-\rho} = 1$ and $\sum a_k^{-\sigma} \log a_k < \infty$, while not all the a_k 's are powers of a_1 . Let $f(n)$ be the number of factorizations of n into the a_k 's, order being taken into account, $f(1) = 1$, $F(n) = \sum f(k)$. Then

$$(1) \quad F(n) = Cn^\rho [1 + o(1)].$$

The proof is unchanged. The author discusses the three alternatives which arise when the assumptions on the a_k 's are not satisfied. (1) If $\sum a_k^{-\rho}$ diverges for all ρ , then $F(n)n^{-\sigma} \rightarrow \infty$ for all σ . [This case requires $a_k = a_{k+1}$ infinitely often which would seem to be excluded by the nature of the problem.] (2) $\sum a_k^{-\rho} < 1$ whenever the series converges. If σ_0 is the abscissa of convergence, then $F(n) = o(n^{\sigma_0})$ and not $O(n^{\sigma_0})$ for any $\sigma < \sigma_0$. (3) There is a ρ such that $\sum a_k^{-\rho} = 1$ but $\sum a_k^{-\sigma} \log a_k = \infty$; here the author can prove merely $\liminf n^{-\sigma} F(n) = 0$. The case in which the a_k 's have all their prime factors in a given finite or infinite set of primes has been studied by E. Hille [Acta Arithmetica 2, 134-144 (1936)] who found formula (1) with $C = \{\rho \sum a_k^{-\rho} \log a_k\}^{-1}$ in generalization of a formula by L. Kalmar for the case $a_k = k$. The author calls attention to the fact that this formula seems to presuppose $\sum a_k^{-\rho} \log a_k < \infty$. In an adden-

dum, p. 651, E. Hille acknowledges the error but points out that his Ikehara-Wiener argument tacitly presupposed $\rho > \sigma_0$, in which case the condition is satisfied, and states that the argument does not apply if either $\rho = \sigma_0$ or $\sum a_k^{-\rho} < 1$ for $\sigma > \sigma_0$. [As reviewer I have to differ from myself as author: the Ikehara-Wiener theorem actually applies in the second case and gives $F(n) = o(n^{\sigma_0})$ in agreement with Erdős's result in case (2) above.] Finally the author acknowledges the priority rights of K. Mahler and K. Knopp to some results in his paper in Ann. of Math. (2) 43, 437-450 (1942) [these Rev. 4, 36]. E. Hille.

Mordell, L. J. On numbers represented by binary cubic forms. Proc. London Math. Soc. (2) 48, 198-228 (1943). [MF 9902]

Let $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ be a binary cubic form with real coefficients a, b, c, d and discriminant $D = 27a^2d^2 - 18abcd - b^2c^2 + 4ac^3 + 4db^3$; the variables x, y are integral and not both zero. Two forms are called equivalent (\sim) if they can be transformed into each other by linear substitutions with integral coefficients. The following theorems are proved. (1) If $D < 0$, then $|f(x, y)| \leq (-D/49)^{1/3}$ is solvable; the equality sign is necessary when and only when $D = -49e^4$ and $f(x, y) \sim e(x^3 + x^2y - 2xy^2 - y^3)$, where e is an arbitrary constant. (2) If $D > 0$, then $|f(x, y)| \leq (D/23)^{1/3}$ is solvable; the equality is necessary when and only when $D = 23e^4$ and $f(x, y) \sim e(x^3 - xy^2 - y^3)$. (3) If $D = 0$, then $|f(x, y)| < e$ is solvable, for arbitrary $e > 0$. In the last case $f(x, y) \sim a(x + py)^2(x + qy)$ with real a, p, q ; therefore the third theorem is an immediate consequence of Minkowski's theorem on linear forms. In the two other cases $f(x, y)$ can be transformed by a real linear substitution into $f_1(x, y) = x^3 + x^2y - 2xy^2 - y^3$ or $f_2(x, y) = x^3 - xy^2 - y^3$, respectively. Let a lattice L be given by $x = \alpha\xi + \beta\eta$, $y = \gamma\xi + \delta\eta$, with arbitrary real $\alpha, \beta, \gamma, \delta$ satisfying $\alpha\delta - \beta\gamma = 1$, where ξ, η run through all integer values. It is proved that a point of L different from $(0, 0)$ lies in each of the regions R defined by $|f_1(x, y)| \leq 1$, $|f_2(x, y)| \leq 1$; a prominent part is played by the linear substitutions which transform R into itself; moreover, Minkowski's theorem is applied to a particular set of parallelograms. C. L. Siegel (Princeton, N. J.).

Todd, J. A. A table of partitions. Proc. London Math. Soc. (2) 48, 229-246 (1943). [MF 9903]

Let $p(n, m)$ denote the number of partitions of n into precisely m parts, or the number of partitions of n into parts of which the largest is m . From the well-known recursion formula of Euler

$$p(n, m) = p(n-1, m-1) + p(n-m, m),$$

we have

$$p(n, m) = \sum_{r=1}^m p(n-m, r),$$

a formula by means of which the author has constructed the present tables of $p(n, m)$ for all m and for $n \leq 100$. [The reviewer has not seen the part beyond $n = 90$.] In case $m \geq n/2$, $p(n, m) = p(n-m)$, where $p(k)$ is the number of unrestricted partitions of k . By tabulating $p(k)$ for $k \leq 50$, the author has saved the printing of half of the table.

The present table is really a considerable extension of Euler's original table of the number (n, m) of partitions of n into parts not exceeding m (for $m \leq 11$ and $n \leq 69$) since $(n, m) = p(n+m, m)$. [L. Euler, Opera (1), Mathematics

The last two pages contain the part of the table for $n > 90$; they appeared only after the main part (cf. the Review's remark in line 10 of the review).

Series, vol. 8, Lausanne, 1922, pp. 327-328]. The obscure table of G. B. Marsano described by Cayley [British Assoc. Advance. Sci. Report 1875, pp. 322-324] and Dickson [History of the Theory of Numbers, vol. 2, Stechert, New York, 1934, p. 127] is only slightly larger, extending as it does to $n=103$. The author states that

$$p(n, m) \sim n^{m-1} / \{m!(m-1)!\}$$

holds for m fixed and $n \rightarrow \infty$. This is in agreement with the recent more precise results of Erdős and Lehner [Duke Math. J. 8, 335-345 (1941); these Rev. 3, 69] and of Gupta [Proc. Indian Acad. Sci., Sect. A. 16, 101-102 (1942); J. Univ. Bombay 11, 16-18 (1942); these Rev. 4, 190, 241].
D. H. Lehmer (Berkeley, Calif.).

Turing, A. M. A method for the calculation of the zeta-function. Proc. London Math. Soc. (2) 48, 180-197 (1943). [MF 9901]

Let L_μ denote the straight line $z = \mu - \sigma r$, where μ is a given positive real number, $\sigma = e^{\pi i/4}$ and r runs through real values from $-\infty$ to $+\infty$; define

$$h(z) = \frac{e^{\pi i z} z^{-s}}{e^{\pi i z} - e^{-\pi i z}}$$

$J(s) = \int_{L_\mu} h(z) dz$, $0 < \mu < 1$, $f(s) = \pi^{-s/2} \Gamma(s/2) J(s)$. Riemann found the formula $\pi^{-s/2} \Gamma(s/2) \zeta(s) = f(s) + \bar{f}(1-s)$ and obtained an asymptotic series for the zeta-function by using saddle-point integration. However, the estimation of the remainder terms is rather complicated unless the imaginary

part t of s is sufficiently large. The author describes a method of calculation of $\zeta(s)$ which is applicable for all values of s . Let m be a positive integer, κ a positive real parameter, $m < \mu < m+1$, $g_1(z) = h(z)(1 - e^{-2\pi i \kappa(z-\mu)})^{-1}$, $g_2(z) = h(z) - g_1(z)$. Then, by Cauchy's theorem,

$$J(s) - \sum_{n=1}^m n^{-s} = \int_{C_1} h(z) dz + \int_{C_2} g_1(z) dz + \int_{C_3} g_2(z) dz,$$

where C_1 and C_2 are certain curves of integration and R is a sum of residues. If μ, κ, C_1, C_2 are suitably chosen, depending upon s , then the two integrals on the right-hand side represent only small terms; their estimation is explicitly given. It is stated that this method is practical for the computation of $\zeta(s)$ in the strip $30 < t < 1000$.

C. L. Siegel (Princeton, N. J.).

Wintner, Aurel. On an harmonic analysis of the irregularities in Goldbach's problem. Revista Ci., Lima 45, 175-182 (1943). [MF 10061]

Supplementing Goldbach's conjecture, Sylvester has surmised that the number of decompositions of an even integer n into a sum of two primes is asymptotically $(An/\log^2 n)f(n)$, where

$$f(n) = \prod_{q|n} ((q-1)/(q-2)).$$

The author finds that $f(n)$ is substantially a multiplicative function and he points out that by previous results of his and others it is almost periodic B^A for arbitrary large λ . There is a similar result for decomposition in r primes, $r > 2$.
S. Bochner (Princeton, N. J.).

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Erdős, P. Some remarks on set theory. Ann. of Math. (2) 44, 643-646 (1943). [MF 9402]

This paper contains four disconnected remarks.

(I) Let R be the set of all real numbers; let $\mathfrak{B}(\mathbb{C})$ be the class of all subsets $s(C)$ of R which are of zero measure (of the first category). W. Sierpiński [Fund. Math. 22, 276-280 (1934)], in explanation of the "duality" between category and measure, proved that, if $2^{\aleph_0} = \aleph_1$, then there is a one-to-one transformation f of R into itself such that $f(C) \in \mathfrak{B}$, $f^{-1}(s) \in \mathbb{C}$; he also stated the problem as to whether there were such functions f which also had the property that $f(s) \in \mathbb{C}$, $f^{-1}(C) \in \mathfrak{B}$. Erdős, again assuming $2^{\aleph_0} = \aleph_1$, constructs such a function.

(II) Let m be a cardinal number. Two sets A, B of a Euclidean space are " m -equivalent" if each can be represented as the sum of m disjoint sets A_i, B_i , where, for all i , A_i and B_i are congruent. The theory of m -equivalence is due to S. Banach and A. Tarski [Fund. Math. 6, 244-277 (1924)], who established, in particular, that any two linear sets containing interior points are \aleph_0 -equivalent. Erdős here constructs 2^c linear sets no two of which are m -equivalent, m being a given cardinal less than c . Lindenbaum had obtained, but not published, this result in the case $m = \aleph_0$.

(III) A theory of absolute measure of linear sets based on "finite-equivalence" is due mainly to Tarski [Fund. Math. 30, 218-253 (1938)]. Erdős, in answering a question raised by Tarski, obtains a new case where this theory deviates from the classical theory. Let $\mathfrak{M}(\mathfrak{M}^*, \mathfrak{B}, \mathfrak{B}^*)$

denote the class of linear sets which are measurable (absolutely measurable, of zero measure, of zero absolute measure). The classes $\mathfrak{M}, \mathfrak{M}^*$ each have cardinal 2^c . It is known that the cardinal of $\mathfrak{M} \bmod \mathfrak{B}$ is c ; it is, however, shown here that the cardinal of $\mathfrak{M}^* \bmod \mathfrak{B}^*$ is 2^c . A result obtained in the course of this proof is used to show that the following result of W. Sierpiński [Fund. Math. 19, 22-28 (1932)] is a best possible one, that is, the c in the conclusion cannot be replaced by a smaller cardinal [this proof is due to P. Lax]: there is a linear set A which, together with its complement, has cardinal c and is such that the intersection of the complement of A and any set congruent to A has a cardinal less than c .

(IV) Let f be a function continuous in the closed interval $(0, 1)$. Let E be the set at which f has its upper right derivate less than $+\infty$. V. Jarník [Fund. Math. 23, 1-8 (1934)] showed that E is not countable. (A very short proof of this, due to A. P. Morse, is included in the present paper.) It is established simply and directly that E has cardinal c .
J. Todd (London).

Miller, Edwin W. A note on Souslin's problem. Amer. J. Math. 65, 673-678 (1943). [MF 9394]

The linear continuum, as an order type, is characterized by the following properties: that it has neither a first nor a last term, that it is continuous, that it contains a countable dense subseries. Souslin [Fund. Math. 1, 223 (1920)] raised the question as to whether the last condition could be replaced by "that any set of nonoverlapping intervals in it should be countable (at most)." In order that the answer to this problem should be in the negative the author

shows that it is necessary and sufficient that there exists a partial order $P, \bar{P} = \mathfrak{N}_1$, such that (a) if $Q \subset P$ and $\bar{Q} = \mathfrak{N}_1$, then Q contains two comparable elements and two incomparable elements, and (b) if x and y are incomparable elements of P then there is no $z \in P$ with $x < z$ and $y < z$. Necessity is readily established; to establish sufficiency appeal is made to results in a paper by the author and Ben Dushnik [Amer. J. Math. 63, 600-610 (1941); cf. these Rev. 3, 73].

J. Todd (London).

Zaubek, Othmar. Über nicht messbare Punktmengen und nicht messbare Funktionen. Math. Z. 49, 197-218 (1943). [MF 10035]

The structure of nonmeasurable sets and functions in a metric space with Carathéodory outer measure is analyzed in terms of such notions as local measurability, local separability and their relativizations. The following are typical results. Any separable nonmeasurable set can be represented as the sum of a measurable set and a maximal totally nonmeasurable set. If to each point of a locally separable space corresponds a set that is nonmeasurable there, there exists a set that is nonmeasurable at every point of the space.

J. C. Oxtoby (Bryn Mawr, Pa.).

Choquet, Gustave. Préliminaires à une nouvelle définition de la mesure. C. R. Acad. Sci. Paris 215, 52-54 (1942). [MF 9478]

The sets considered are subsets of a given set E . A relation between certain pairs of subsets is called a congruence if it is reflexive, symmetric and transitive. A definition of the "equality of mass" of two sets is called a definition Δ provided a set of five axioms is verified. The set of all definitions Δ forms a partially ordered system with order being given a natural definition. To avoid certain difficulties a subfamily of definitions δ of the preceding family is considered. This subfamily is defined in terms of certain transformations as well as a certain notion of congruence. There exist definitions δ which are smaller and larger than all the others.

D. Montgomery (Princeton, N. J.).

Choquet, Gustave. Choix d'une mesure cartésienne Δ . Applications. C. R. Acad. Sci. Paris 215, 101-103 (1942). [MF 9480]

This note considers the choice of a Δ which will be a δ [see the preceding review] for a Cartesian space where congruence is defined by means of ordinary displacement. The choice is made by the use of isometries. An isometry between A and A' is a homeomorphism such that for every M of A if P tends toward M the ratio of MP to $M'P'$ tends toward 1. Two isometric sets have the same Lebesgue measure if at least one of them is Lebesgue measurable. Applications are made to sets applicable one on the other, to total variation and to integrals.

D. Montgomery.

Kantorovitch, L. On the translocation of masses. C. R. (Doklady) Acad. Sci. URSS (N.S.) 37, 199-201 (1942). [MF 8660]

The following problem is considered. Suppose there are given two mass distributions $\phi(e), \phi'(e')$ (that is, two non-negative additive set functions) over a compact metric space, and a continuous non-negative function $r(x, y)$ which represents the work expended in transferring a unit mass from point x to point y . The problem is to transport the original mass distribution $\phi(e)$ into $\phi'(e')$ in such a way

as to require a minimum expenditure of work. A necessary and sufficient condition is obtained for such a minimal transportation, which at the same time provides an effective process for reducing the work if the transportation is not minimal. Applications are suggested to several practical problems. The proofs are very sketchy, and numerous misprints occur.

M. Shiffman (New York, N. Y.).

Miller, Donald S. Carathéodory and Gillespie linear measure. Duke Math. J. 11, 139-145 (1944). [MF 10154]

A. P. Morse and J. F. Randolph [Duke Math. J. 6, 408-419 (1940); these Rev. 1, 304] showed that the Gillespie linear measure $G(A)$ of a plane point set A is related to its Carathéodory linear measure $L(A)$ by the inequalities $L(A) \leq G(A) \leq (\pi/2)L(A)$, both bounds being the best possible. The lower equality holds for any rectifiable set and it was asserted without proof that a certain closed set constructed by Besicovitch [Math. Ann. 98, 422-464 (1927); in particular, p. 431] for a different purpose realizes the upper equality. The present paper contains a proof of the latter assertion.

J. C. Oxtoby (Bryn Mawr, Pa.).

Ward, A. J. The Fan integrals interpreted as measures in a product-space. Amer. J. Math. 66, 144-160 (1944). [MF 9946]

Let E_0 be a set on the x -axis for which a measure function m has been defined and $(0, M)$ an interval on the positive y -axis. Let E be any set in the Cartesian product space $E_0 \times (0, M)$. Let E be covered by a sequence of sets of the form $E_n \times J_n$, where E_n is a subset of E_0 and J_n is a Lebesgue measurable set on $(0, M)$. Then mE is the lower bound of $\sum mE_n \cdot |J_n|$ for all such coverings. The lower measure of the set E is defined as

$$m[E_0 \times (0, M)] - m[(E_0 \times (0, M)) - E].$$

If E is the ordinate set of an arbitrary bounded function $f(x)$, then the integrals of $f(x)$ defined by Fan are expressed in terms of the upper or lower measure of the set E . This approach permits some of the results of Fan to be obtained in a more simple manner.

R. L. Jeffery.

Schwartz, H. M. Sequences of Stieltjes integrals. III. Duke Math. J. 10, 595-610 (1943). [MF 9742]

[The first two parts appeared in Bull. Amer. Math. Soc. 47, 947-955 (1941); Duke Math. J. 10, 13-22 (1943); cf. these Rev. 3, 228; 4, 154.] If V^*g is the variation of any function g^* agreeing with the function of bounded variation g at all points of continuity but having no external saltus, then the principal theorem of the paper is the following. If $g_n(x)$ are of bounded variation on (a, b) with $g_n(a) = 0$, then necessary and sufficient conditions that $\int_a^b f(x) dg_n(x) \rightarrow 0$ for all $f(x)$ for which all of the integrals exist are that (1) $g_n(x) \rightarrow 0$ for all common continuity points of the g_n , (2) V^*g_n be bounded and (3) $V_K^*g_n = 0$ uniformly for every closed set K for which $V_Kg_n = 0$, V_Kg being the variation of g on the set K , and $V_Kg_n = 0$ uniformly meaning that for every $\epsilon > 0$ there exists a subdivision D_ϵ independent of n such that $\sum_{D_\epsilon} \Delta V g_n < \epsilon$, D_ϵ being the intervals of D_ϵ whose closure intersects K . This gives rise to conditions so that $\int_a^b f dg$ exists and equals $\lim_n \int_a^b f dg_n$, where $g(a) = 0$ and $g(x)$ has no external saltus. Application is made to the case where $g_n(x) = \int_a^x G_n(t) dt$, $G_n(t)$ being Lebesgue integrable on (a, b) , connecting with Lebesgue's results [Ann. Fac. Sci. Toulouse (3) 1, 25-117 (1909)] for the same situation.

T. H. Hildebrandt (Ann Arbor, Mich.).

Singh, A. N. On functions without one-sided derivatives. Proc. Benares Math. Soc. (N.S.) 3, 55-69 (1941). [MF 9448]

In part I of this paper there is an analytic definition of the function without one-sided derivatives which was defined by Besicovitch [Bull. Acad. Sci. Russie (2) 19, 527-540 (1925)]. This definition enables him to make a close study of the derivatives at any given point. He shows that there are sets S_1 : $0 < D^+ < \infty$, $D_+ \leq 0$, $D^- \neq D_- > 0$; S_2 : $D^+ \geq 0$, $-\infty < D_+ < 0$, $D_- \neq D^- < 0$; S_3 : $D^+ \neq D_+ > 0$, $0 < D^- < \infty$, $D_- \leq 0$; S_4 : $D_+ \neq D^+ < 0$, $-\infty < D_- < 0$, $D^- \geq 0$; S_5 : one of the extreme derivatives on the right (left) is infinite and a median or extreme derivative is zero. In part II, to appear in the next volume of the same Proceedings, the author claims to give a simple method of constructing an infinite class of functions without one-sided derivatives.

R. L. Jeffery (Kingston, Ont.).

Zygmund, A. A theorem on generalized derivatives. Bull. Amer. Math. Soc. 49, 917-923 (1943). [MF 9690]

A function $f(x)$ defined in the neighborhood of a point x_0 is said to possess a k th generalized derivative at this point if

$$f(x_0+t) = a_0 + a_1 t + a_2 t^2/2! + \dots + a_{k-1} t^{k-1}/(k-1)! + \omega_k(t) t^k/k!,$$

where a_0, a_1, \dots, a_{k-1} are independent of t and $\omega_k(t) = \omega_k(x_0, t)$ approaches a finite limit α_k as $t \rightarrow 0$. The k th generalized derivative is $\alpha_k = D_k f(x_0)$: its existence implies that of $D_{k-1} f(x_0)$. The following theorem is proved. Suppose that a function $f(x)$ is of class L^2 and of period 2π and that the generalized derivative $D_k f(x)$ exists for every point x of a set E of positive measure ($k=1, 2, \dots$). Let $\omega_k(x, t)$ and $e_k(x, t)$ be defined by the equations

$$f(x+t) = \sum_{j=0}^{k-1} D_j f(x) t^j/j! + \omega_k(x, t) t^k/k!$$

$$\omega_k(x, t) = D_k f(x) + e_k(x, t).$$

Then the integral

$$\int_0^{2\pi} \frac{[\omega_k(x, t) - \omega_k(x, -t)]^2}{t} dt = \int_0^{2\pi} \frac{[e_k(x, t) - e_k(x, -t)]^2}{t} dt$$

is finite for almost every $x \in E$. This theorem includes a result of Marcinkiewicz [Ann. Soc. Polon. Math. 17, 42-50 (1938)] to which it reduces for $k=1$.

R. Salem.

Theory of Functions of Complex Variables

Vigil, Luis. On Taylor series with, or without, analytic continuation. The present state of Borel's theorem. Revista Mat. Hisp.-Amer. (4) 3, 208-218 (1943). (Spanish) [MF 10160]

Conclusion of an expository paper; the first part appeared in the same Revista (4) 3, 137-144 (1943); cf. these Rev. 5, 35.

R. P. Boas, Jr. (Cambridge, Mass.).

Sunyer Balaguer, F. Supplementary note to the article: On some results concerning the theorems of Picard, Landau and Schottky and on a criterion of quasi-normality. Revista Mat. Hisp.-Amer. (4) 3, 206-207 (1943). (Spanish) [MF 10140]

The article referred to appeared in the same Revista (4) 2, 88-96, 271-278 (1942); cf. these Rev. 4, 241.

R. P. Boas, Jr. (Cambridge, Mass.).

Tschebotarow, N. On Hurwitz's problem for transcendent functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 479-481 (1941). [MF 9633]

An entire function $f(z) = \sum a_n z^n$ with a_n real ($n \geq 0$) is said to be an H -function if the real parts of the roots of $f(z) = 0$ are negative. The author proves that, if $f(z)$ is of order 0 and if $\phi(z) = \frac{1}{2}[f(z) + f(-z)]$, $\psi(z) = \frac{1}{2}[f(z) - f(-z)]$, then $f(z)$ is an H -function if and only if (1) the roots of $\phi(z) = 0$ are purely imaginary, (2) $\psi(\alpha)/\phi'(\alpha) > 0$ holds for every root α of $\phi(z) = 0$. These conditions are also sufficient for a function of order 1.

S. Mandelbrojt.

Macphail, M. S. Entire functions bounded on a set. Trans. Roy. Soc. Canada. Sect. III. 37, 31-38 (1943). [MF 9975]

Let $\phi(z)$ be an entire function of order 1 with infinitely many simple zeros z_n and satisfying a certain set of eight conditions. (Lack of space prevents reproducing them.) If $f(z)$ is an entire function of minimal type of order 1 and $f(z_n)/\phi'(z_n) = O(|z_n|^{-\lambda})$, then $f(z) = 0$. In case $\phi(z) = \sin \pi z$ this furnishes a slightly sharper formulation of a theorem of Pólya. The author considers various examples, in particular, certain almost periodic functions and some Bessel functions.

G. Szegő (Stanford University, Calif.).

Boas, R. P., Jr. Functions of exponential type. I. Duke Math. J. 11, 9-15 (1944). [MF 10142]

Let $M[f(x)]$ denote the limit of the average of $f(x)$ over $(-T, T)$ as $T \rightarrow \infty$. The author proves that, if $f(s)$ is an entire function of exponential type C such that $M[|f(x)|]$ exists, then $M[f(x)e^{i\lambda x}]$ exists and is zero for real λ , $|\lambda| > C$. This generalizes the known result that a periodic entire function of exponential type is a finite trigonometric sum. Because the proof is very simple if $f(x)$ is bounded the author shows that his result is not vacuous by giving an example of an unbounded $f(x)$ which is Besicovitch almost periodic on the real axis (and for which therefore $M[|f(x)|]$ must exist).

N. Levinson (Cambridge, Mass.).

Boas, R. P., Jr. Functions of exponential type. II. Duke Math. J. 11, 17-22 (1944). [MF 10143]

[Cf. the preceding review.] Whittaker's problem consists of finding the largest number W such that, if $f(s)$ is an entire function of exponential type C every derivative of which has at least one zero in the unit circle, then $C < W$ implies that $f(s) = 0$. It has been known that $.693 = \log 2 \leq W \leq \frac{1}{2}\pi = .785$. In an unpublished result Pólya has shown $W > \log 2$. The author shows that $.718 < W < .748$. [The author wishes to retract the statement on p. 20 that the smallest roots of the function f_s exceeds .84, and that that of f_n becomes infinite with n .]

N. Levinson.

Schaeffer, A. C. and Spencer, D. C. The coefficients of schlicht functions. Duke Math. J. 10, 611-635 (1943). [MF 9743]

Let $F(E)$ be the family of functions which are regular and schlicht in the interior of the unit circle E and normalized at the origin so that each function has the value zero and its first derivative the value one. Let C be the set of domains into which the unit circle is mapped by functions of $F(E)$. Let $F(G)$ be the family of functions

$$f(z) = z + a_2 z^2 + \dots + a_n z^n + \dots$$

regular and schlicht for $z \in G$, where G is a domain of C . Let $\alpha_n = \alpha_n(G) = \sup f \in F(G) | a_n |$, $\gamma_n = \inf \alpha_n(G)$, $\Gamma_n = \sup \alpha_n(G)$. It has been widely conjectured that $\alpha_n(E) = n$ and the con-

jecture is known to be true for $n=2$ and 3. Previously Löwner had given the only proof for $n=3$. The authors give new proofs of $\alpha_2(E)=2$, $\alpha_3(E)=3$. They show also that $\alpha_n(E)=\gamma_n$, and that if $\alpha_n(E)=n$ is true then $\Gamma_n=4^{n-1}$. For the odd schlicht functions it is known that the coefficients are bounded. In establishing the existence of an odd schlicht function with the extremal fifth coefficient equal to $e^{-1} + \frac{1}{2}$, M. Fekete and G. Szegő did not exhibit the function. The authors have succeeded in characterizing this unique function as $\phi(z) = (f(z^2))^{\frac{1}{2}}$, where $f(z)$ is the function w defined by the equation

$$-\frac{(1-\alpha w)^{\frac{1}{2}}}{w} + \frac{\alpha}{2} \log \frac{1+(1-\alpha w)^{\frac{1}{2}}}{1-(1-\alpha w)^{\frac{1}{2}}} = -\frac{1}{z} - \frac{\alpha}{2} \log z + z$$

with $\alpha=4e^{-1}$. This function has real coefficients. The authors further show that for any $n>1$ there are odd schlicht functions with real coefficients for which the $(2n+1)$ st coefficient exceeds one in absolute value.

These interesting results are consequences of the authors' principal result that the function $f(z)$, regular and schlicht in E with $|a_n| = \alpha_n(E)$, satisfies the differential equation

$$[zf'(z)]^{\frac{1}{2}} \sum_{n=1}^{\infty} a_n^{(*)} f(z)^{-n-1} = n-1 + \sum_{n=1}^{\infty} \left(\frac{va_n}{a_n} z^{-n+r} + \frac{va_n^*}{a_n^*} z^{-n-r} \right).$$

The $a_n^{(*)}$ are defined by $f(z)^* = \sum_{n=1}^{\infty} a_n^{(*)} z^n$, and a^* denotes the complex number conjugate to a . The authors also obtain an analogous differential equation for the extremal functions of $F(G)$. These equations were obtained by new variational methods. Since the boundary of $w=f(z)$ with $|a_n| = \alpha_n(E)$ is known to consist of analytic curves [M. Schiffer, Proc. London Math. Soc. (2) 44, 432-449 (1938)], there is a closed arc of the unit circle in a complete neighborhood of which $f(z)$ is analytic. A composite mapping is set up corresponding to infinitesimal variations of this arc. The resulting normalized schlicht function has an n th coefficient of absolute value necessarily smaller than $\alpha_n(E)$. This fact results in the above differential equation.

The reviewer observes that the theorem includes as a special case the equality of F. Marty [C. R. Acad. Sci. Paris 198, 1569-1571 (1934)]

$$2a_2 a_n^* = (n+1)a_{n+1} - (n-1)a_{n-1}, \quad |a_n| = \alpha_n(E),$$

which was obtained by a different variational method.

M. S. Robertson (New Brunswick, N. J.).

Rosenblatt, Alfred. On power series in the unit circle. Revista Ci., Lima 45, 195-225 (1943). (Spanish) [MF 10062]

For bounded analytic functions $f(z)$, with $|f(z)| \leq 1$ for $|z| < 1$, the author first recalls several inequalities involving the coefficients a_n , the partial sums of the coefficients and the majorant $M(\rho) = \sum_{n=0}^{\infty} |a_n| \rho^n$. He then develops inequalities analogous to some of the above for analytic functions $f(z)$ with bounded first derivatives, $|f'(z)| \leq 1$ for $|f(z)| < 1$; for example, he shows that

$$\sum_{n=1}^{\infty} |a_n| \leq \pi/\sqrt{6}.$$

Functions which map the interior of the unit circle on domains of finite area A also are studied. A typical result is the following: the inequality of Gronwall

$$M(\rho) \leq \left(\frac{A}{\pi} \log \frac{1}{1-\rho^2} \right)^{\frac{1}{2}}$$

now is supplemented by

$$M(\rho) = o \left(\log \frac{1}{1-\rho} \right)^{\frac{1}{2}}.$$

At the end of the article is a list of 44 inequalities.

E. F. Beckenbach (Austin, Tex.).

Claus, Heinrich. Neue Bedingungen für die Nichtfortsetzbarkeit von Potenzreihen. Math. Z. 49, 161-191 (1943). [MF 10054]

The author generalizes classical theorems of Hadamard and Fabry concerning Taylor series admitting the circle of convergence as a cut. Let $\phi(x)$ ($x \geq 0$) be a positive differentiable function such that $\phi'(x)$ decreases monotonically to zero and such that $\phi(x)/x \rightarrow 0$, $\phi(x)/\log x \rightarrow \infty$ as $x \rightarrow \infty$. If (a) $\limsup |a_n|^{1/\phi(n)} = 1$, (b) there exists a subsequence $\{n_i\}$ such that $\lim |a_{n_i}|^{1/\phi(n_i)} = 1$ and such that, for the set of all the integers n satisfying one of the two inequalities $n_i - \phi(n_i) \leq n < n_{i+1}$ ($i=1, 2, \dots$) or $n_i < n \leq n_{i+1} + \phi(n_i)$, the relationship $\limsup_{i \rightarrow \infty} |a_n|^{1/\phi(n_i)} < 1$ holds, then the circle of convergence (which is of radius one) of $\sum a_n z^n$ is a cut. Generalizations of this theorem are given.

S. Mandelbrojt (Houston, Tex.).

Piranian, George. On the convergence of certain partial sums of a Taylor series with gaps. Bull. Amer. Math. Soc. 49, 881-885 (1943). [MF 9683]

The author establishes an interesting extension of Ostrowski's well-known theorem on overconvergence [S.-B. Preuss. Akad. Wiss. 1921, 557-565 (1921)]. Let $f(z) = \sum c_p z^{p_n}$ be a power series with unit radius of convergence and write $S_n = \sum c_p z^{p_n}$, $M(r) = \max_{|z|=r} f(z)$ and $\theta_n = \lambda_{n+1}/\lambda_n - 1$. The author proves that, if

$$\limsup_{i \rightarrow \infty} \frac{\log \{M(1-\theta_{n_i})/\theta_{n_i}\}}{\lambda_{n_i} \theta_{n_i}} < \infty,$$

then $\lim S_n(z) = f(z)$ at all regular points of $f(z)$ on $|z|=1$.

A. C. Offord (Newcastle-on-Tyne).

Mandelbrojt, S. Quasi-analyticity and analytic continuation—a general principle. Trans. Amer. Math. Soc. 55, 96-131 (1944). [MF 9877]

The author gives a general theorem providing estimates for coefficients d_n under the assumption that a bounded analytic function $F(s)$ is represented asymptotically, in a very general sense, by a sequence of exponential sums $\sum d_n e^{-\lambda_n s}$; the estimates depend on the rate of growth of the sequence $\{\lambda_n\}$, on the character of the asymptotic representation and on the maximum modulus of the function on suitable circles. A number of lemmas allow one to infer that hypotheses of the kind which enter naturally into the applications imply the hypotheses of the general theory. As applications, the author obtains new results of great generality on Watson's problem on the uniqueness of a function with a given asymptotic expansion, and on quasi-analytic functions. An application in a different direction is to the analytic continuation of a function represented asymptotically by exponential sums, and consequently to the analytic continuation of functions defined by Dirichlet series; Fabry's gap theorem on power series appears as a very special case.

It is not possible to summarize adequately in a short review the numerous significant advances represented by this paper. [Erratum: on p. 126, in theorem 7, one should read $2\pi a$ for $2\pi a$ in the argument of \sum (twice), and "one" for "two" in conclusion c.] R. P. Boas, Jr.

Stollow, S. Sur les singularités des fonctions analytiques multiformes dont la surface de Riemann a sa frontière de mesure harmonique nulle. *Mathematica, Timișoara* 19, 126-138 (1943). [MF 9960]

This paper is concerned with functions which are meromorphic on a Riemann surface whose boundary is of harmonic measure zero. Its object is to relate Pompeiu's earlier investigations [*Ann. Fac. Sci. Toulouse* (2) 7, 265-315 (1905)] on uniform analytic functions whose singularities form a set of linear measure zero with the investigations of the author [*Mathematica, Timișoara* 12, 123-138 (1936)] on meromorphic functions which satisfy the conclusion of Iversen's theorem. By means of R. Nevanlinna's principle of harmonic measure it is shown that Iversen's theorem persists for functions which are meromorphic on a Riemann surface with boundary of harmonic measure zero. Hence the above investigations of Stollow permit the conclusion that a function meromorphic on a Riemann surface with boundary of harmonic measure zero either has every complex number as a limit value relative to the boundary of its Riemann surface or else its inverse is a generalized algebraic function. Refinements of this result are given.

M. H. Heins (Chicago, Ill.).

Lelong, Pierre. Sur une propriété de la frontière d'un domaine d'holomorphie. *C. R. Acad. Sci. Paris* 216, 107-109 (1943). [MF 10003]

A function is said to be multi-subharmonic in a domain D if it is such that (a) it has a definite real value, finite or equal to $-\infty$, at each point of D ; it takes on a finite value in D at least once; it has a finite upper limit in the interior of D ; (b) in each analytic complex plane, the function is subharmonic or is identically equal to the constant $-\infty$. Let C^n be a space of n complex variables z_k ($k=1, 2, \dots, n$). Let $M(z_1^0)$ be a point inside a domain D with boundary F . The plane $z_k = z_k^0 + a_k a$ drawn through M parallel to the direction (a_k) cuts F in a nonempty closed set \mathcal{E} . The distance of M from F parallel to the direction (a_k) is defined as the distance of M from \mathcal{E} . If the power series

$$f(z_1, z_2, \dots, z_n) = \sum_n (z_1 - z_1^0)^n A_n(z_1^0, z_2, \dots, z_n)$$

converges for $|z_1 - z_1^0| < R(z_1^0, z_2, \dots, z_n)$, then $-\log R$ is a function multi-subharmonic in the variables z_1^0, z_2, \dots, z_n . Thus a necessary condition that a domain be a domain of holomorphy is that $-\log \delta(M, a_k)$ be a function multi-subharmonic of M , where $\delta(M, a_k)$ denotes the distance of an interior point M of D to its boundary whatever be the direction (a_k) . Some applications of this enunciation are made.

M. S. Robertson (New Brunswick, N. J.).

Lelong, Pierre. Sur la capacité de certains ensembles de valeurs exceptionnelles. *C. R. Acad. Sci. Paris* 214, 992-994 (1942). [MF 9473]

The author draws some obvious conclusions of theorems given by H. Cartan [*C. R. Acad. Sci. Paris* 214, 994-996 (1942); these *Rev.* 5, 146] for the theory of sequences of analytic functions. Most notable are some applications to the theory of functions of two complex variables: the necessary and sufficient condition that $f(x, y)$, analytic in a neighborhood of $x=0, y=0$ should be an entire function of y for all $x \in E, y=0$, is that E should have exterior capacity zero.

František Wolf (Berkeley, Calif.).

Dufresnoy, Jacques. Esquisse d'une théorie des familles complexes normales. *C. R. Acad. Sci. Paris* 216, 681-683 (1943). [MF 9991]

A modification of Montel's definition of a normal complex family (famille complexe normale) suggested by the work of H. and J. Weyl and Ahlfors on meromorphic curves is proposed. The theory of such families based upon the proposed definition is sketched in broad outline.

M. H. Heins (Chicago, Ill.).

Dufresnoy, Jacques. Sur les familles complexes normales. *C. R. Acad. Sci. Paris* 216, 715-717 (1943). [MF 9993]

The outline given in the paper reviewed above is supplemented by further details and applications to theorems of the Landau-Schottky type for algebraic functions.

M. H. Heins (Chicago, Ill.).

Martin, W. T. Mappings by means of systems of analytic functions of several complex variables. *Bull. Amer. Math. Soc.* 50, 5-19 (1944). [MF 9871]

Der Verfasser gibt einen Überblick über die bisherigen Ergebnisse, die bei der Behandlung der beiden folgenden Probleme aus der Theorie der Abbildungen im Raume n komplexer Veränderlichen erhalten wurden: 1). Die Aufstellung allgemeiner Kriterien, die zu entscheiden gestatten, ob zwei gegebene Bereiche des Raumes R_n durch ein System von n analytischen Funktionen von je n komplexen Veränderlichen aufeinander abgebildet werden können oder nicht. 2). Die Angabe eines vollständigen Systems von "Repräsentantenbereichen" des R_n mit der Eigenschaft, dass jeder vorgegebene Bereich auf einen dieser Repräsentantenbereiche abgebildet werden kann. Dem Bericht liegen die diesbezüglichen Arbeiten von Behnke, Behnke-Peschl, Bergmann, Carathéodory, H. Cartan und Welke zu Grunde, ausserdem das Manuskript eines Buches von Bochner-Martin und insbesondere der Abschnitt über die Abbildungstheorie in Behnke-Thullen: Theorie der Funktionen mehrerer komplexer Veränderlichen [*Ergebnisse der Mathematik* 3, no. 3, Springer, Berlin, 1934]. Zum ersten Problem betrachtet der Verfasser vornehmlich die "inneren" Abbildungen und die Automorphismen eines Bereiches, insbesondere die Cartanschen Sätze über die Gruppen von Automorphismen, die einen inneren Punkt festlassen. Ausgangspunkt sind die Eindeutigkeitsätze von Carathéodory und Cartan. Ausser den bisher bekannten Ergebnissen gibt der Verfasser einige interessante Verallgemeinerungen und Erweiterungen an. Bezüglich des zweiten Problems werden die Hauptpunkte der Bergmannschen Abbildungstheorie entwickelt.

P. Thullen (Quito, Ecuador).

Peschl, E. Über den Cartan-Carathéodoryschen Eindeutigkeitsatz. *Math. Ann.* 119, 131-139 (1943). [MF 10096]

An important theorem in the theory of mappings of analytic functions of several complex variables is the following result due to Cartan and Carathéodory. "Let D be a bounded domain in E_n containing the origin and let (*) $T: z_j' = f_j(z_1, \dots, z_n) = z_j + \text{higher powers}, j=1, \dots, n$, be an inner mapping of $D, TD \subset D$. Then T is the identity: $f_j(z) = z_j$." The result is not true for general unbounded domains. In spite of this Behnke and Peschl [*Math. Ann.* 114, 67-73 (1937)] were able to prove a generalization of it to certain unbounded domains. In the present paper Peschl generalizes the result still further. He proves: "Let D be a domain in E_n containing the origin and let there exist n bounded analytic functions $G_j(z_1, \dots, z_n)$ in D with

a not identically vanishing functional determinant. Then every inner mapping of the form (*) is the identity." (If D is not schlicht it must be unbranched at the origin.) The proof uses the diagonal expansions of the function G , but in a somewhat different manner from that of the earlier work.

W. T. Martin (Syracuse, N. Y.).

Roure, Henri. Sur une généralisation de la série de Lagrange, avec applications à l'astronomie. C. R. Acad. Sci. Paris 216, 332-338 (1943). [MF 10016]

Describes Lagrange's expansion for several variables and its application to problems on two planets entirely analogous to the one variable case.

P. Franklin.

Fourier Series and Generalizations, Integral Transforms

Szász, Otto. On some trigonometric summability methods and Gibbs' phenomenon. Trans. Amer. Math. Soc. 54, 483-497 (1943). [MF 9521]

This paper is a sequel to an earlier paper of the author [Trans. Amer. Math. Soc. 53, 440-453 (1943); these Rev. 4, 244]. He defines the generalized jump of an odd function $f(t)$ at $t=0$ as

$$(*) \quad \lim_{\theta \rightarrow 0} (2(k+1)/\theta^{k+1}) \int_0^\theta (\theta-t)^k f(t) dt = j > 0$$

and the Fourier series of $f(\theta)$ presents a generalized Gibbs phenomenon at $\theta=0$ if $\limsup s_n(\theta_n) > j/2$, where

$$(**) \quad s_n(\theta) = \sum_{\nu=1}^n b_\nu \sin \nu \theta.$$

The author first studies the means

$$T_n(\theta_n) = \sum_{\nu=1}^n \tau_\nu ((\sin \nu \theta_n)/\nu)$$

for given sequences $\{\tau_\nu\}$ and obtains conditions for the convergence to a limit of (i) $\frac{1}{2} \{T_n(\theta_n) + T_n(\phi_n)\}$, (ii) $(1/n) \sum_{\nu=1}^n T_\nu(\theta_n)$, (iii) $(1/\theta_n) \int_0^{\theta_n} (\sum_{\nu=1}^n \tau_\nu \nu^{-1} \sin \nu t) dt$. He then applies these results to determine the jump j in terms of the behavior of the partial sums (**). He also generalizes Rogosinski's result [Schriften Königsberg. Gel. Ges. 3, 57-98 (1926)] that the length of the Gibbs set is at least j/g , where $g = (2/\pi) \int_0^\pi t^{-1} \sin t dt$. Szász shows that this result remains true if $f(t)$ satisfies (*) with $k=0$ and a supplementary condition.

A. C. Offord (Newcastle-on-Tyne).

Guinand, A. P. Fourier series and primitive characters. Quart. J. Math., Oxford Ser. 14, 79-81 (1943). [MF 9933]

If $k > 1$ and $\chi(n)$ is a primitive character modulo k , $\bar{\chi}(n)$ is the conjugate of $\chi(n)$, and

$$\tau(k, \chi) = \sum_{n=1}^k \chi(n) e^{2\pi i n/k},$$

then a function $F(x)$ has the generalized Fourier series

$$\bar{\chi}(m) F(x) = \sum_{n=-\infty}^{\infty} c_n \chi(n) e^{2\pi i n(x+m)/k},$$

$$c_n = (1/\tau(k, \chi)) \int_{-1}^1 e^{-2\pi i n t/k} F(t) dt.$$

The proof consists essentially in expanding $F(x)$ in an ordinary Fourier series and rearranging the product $F(x) \sum_{r=1}^k \chi(r) e^{2\pi i r m/k}$; consequently, as the author observes,

any standard conditions for the convergence of a Fourier series would apply; he actually uses continuity plus bounded variation. For real primitive characters, the series takes the simpler form

$$\chi(m) F(x) = \sum_{n=1}^{\infty} \chi(n) \left\{ a_n \frac{\cos [2\pi n(x+m)/k]}{\sin [2\pi n(x+m)/k]} \pm \frac{\sin [2\pi n(x+m)/k]}{\cos [2\pi n(x+m)/k]} \right\},$$

where the upper or lower functions and sign are taken according as $\chi(-1) = +1$ or -1 , and

$$a_n = (2/k!) \int_{-1}^1 F(t) \frac{\cos (2\pi n t/k)}{\sin (2\pi n t/k)} dt.$$

R. P. Boas, Jr. (Cambridge, Mass.).

Levine, B. Sur la constant séculaire d'une fonction holomorphe presque périodique. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 182-185 (1941). [MF 9627]

Let $f(z) \sim \sum a_n e^{i \lambda_n z}$ be an analytic almost periodic function and let its Fourier exponents λ_n have a lower bound Δ . Let $\mu(y)$ be the secular constant of $f(x+iy)$; that is, let

$$\mu(y) = \lim_{x \rightarrow \infty} \frac{\arg f(x+iy)}{x},$$

where $\arg f(x+iy)$ is continuous for each fixed x except when passing zeros of $f(z)$, and $-\pi < \arg f(iy) \leq \pi$. Then the author shows that $\lim_{y \rightarrow \infty} \mu(y) = \Delta$. Moreover, if $\mu(y)$ exists [Hartmann has shown that the set of values of y for which it fails to exist is denumerable] and if $n(t, y)$ denotes the number of zeros of $f(z)$ in the rectangle $y_1 \leq y \leq y_1+t$, $|x| \leq t$, the author proves that

$$\lim_{t \rightarrow \infty} (n(t, y_1)/t) = (1/\pi) [\mu(y_1) - \Delta].$$

R. H. Cameron (Cambridge, Mass.).

Rumschisky, L. Über einige Klassen von positiv-definiten Funktionen. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 105-108 (1941). [MF 9623]

Denote by P_A the class of measurable functions in $-A < t < A$ which can be represented by $\int_{-\infty}^{\infty} e^{it\alpha} d\sigma(\alpha)$, with $d\sigma(\alpha) \geq 0$, $\int d\sigma(\alpha) < \infty$; let P_A^L be the subclass arising from integrals $\int_{-\infty}^{\infty} e^{it\alpha} \phi(\alpha) d\alpha$, $0 \leq \phi(\alpha) < L$. For finite A these classes have been investigated by Krein and Acheyser, and the author adds to their results. The following statement is typical. If $F(t)$ belongs to P_A^L and if $F^{(n)}(t)$ is the resultant

$$\int_0^t F^{(n-1)}(t-\tau) F(\tau) d\tau,$$

with $F^{(0)}(t) = F(t)$, then

$$M(t) = \sum_{n=1}^{\infty} (i^{n-1}/n! L^n) F^{(n)}(t)$$

belongs to P_A and

$$L^{-1} F(t) = \sum_{n=1}^{\infty} ((-i)^{n-1}/n!) M^{(n)}(t)$$

is the inverse formula. S. Bochner (Princeton, N. J.).

Pollard, Harry. Representation as a Gaussian integral. Duke Math. J. 10, 59-65 (1943). [MF 8101]

The following set of conditions is shown to be necessary and sufficient for $f(x)$ to have the representation

$$f(x) = \int_{-\infty}^{\infty} e^{-t^2} e^{x^2 t^2} d\beta(t), \quad -a < x < a,$$

$$z^n 2^{-n} f^{(2n)}(0)/n!$$

with $\beta(t)$ nondecreasing and bounded: $f(x)$ is analytic in $(-a, a)$; the quadratic forms

$$\sum_{i,j=0}^n f^{(i+j)}(0) x_i x_j \geq 0$$

for $n \geq 0$; $\sum_{n=0}^{\infty} f^{(2n)}(0) x^{2n}/n!$ converges. The chief interest of the paper is in the method of proof, which depends on the positivity of the Abel kernel for Hermite series in a way paralleling Widder's discussion of the one-sided Laplace transform [The Laplace Transform, Princeton University Press, Princeton, N. J., 1941, pp. 168-177; these Rev. 3, 232]. R. P. Boas, Jr. (Cambridge, Mass.).

Kober, H. A note on Hilbert transforms. Quart. J. Math., Oxford Ser. 14, 49-54 (1943). [MF 9931]

The author develops the L^p theory ($p > 1$) of the Hilbert transform

$$\mathfrak{H}f = \frac{1}{\pi} \int_{0+}^{\infty} \frac{f(x+t) - f(x-t)}{t} dt$$

by starting with rational functions and then approximating by them to the general $f(x)$ of L^p . A new result is that $f(x) \in L^p$ and satisfies $\mathfrak{H}f = \pm if$ if and only if $f(x)$ can be approximated in the L^p metric by rational functions vanishing at infinity and regular for $y \geq 0$ ($+$) or $y \leq 0$ ($-$). Further, in the spaces of such characteristic functions, the sequences $\{\pi^{-1/2}(t-i)^n(t+i)^{-n-1}\}$ ($n=0, 1, 2, \dots$ for $+$, $n=-1, -2, \dots$ for $-$) are orthonormal and closed. [For the L^1 theory, cf. Kober, Bull. Amer. Math. Soc. 48, 421-427 (1942); these Rev. 4, 40.] R. P. Boas, Jr.

Taldykin, A. T. Minimal and regular systems of functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 125-128 (1943). [MF 10042]

Various conditions are given for the convergence in the mean of the biorthogonal series expansion of a function of $L^2(a, b)$ in terms of a uniformly minimal and regular system (1) $\{\phi_i\}$ of functions of $L^2(a, b)$. The system (1) is called regular if the Gramian matrix $(\int_a^b \phi_i \phi_j dx)$ is regular. The term uniformly minimal was defined in a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 531-534 (1940); these Rev. 2, 193]. O. Frink.

Taldykin, A. T. Classification of certain systems of functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 170-172 (1943). [MF 10044]

It is shown that a system $\{\phi_i\}$ of functions of $L^2(a, b)$ is (1) uniformly minimal, (2) not minimal or (3) weakly minimal according as the number zero is (1) not in the spectrum, (2) not a proper value of the spectrum or (3) a proper value of the spectrum of the infinite Gramian matrix $(\int_a^b \phi_i \phi_j dx)$ of the system. Other types of systems called partly minimal and uniformly nonminimal are considered.

O. Frink (State College, Pa.).

Polynomials, Polynomial Approximations

Garnea, E. G. On a property of the roots of the derivative of a polynomial of third degree. Revista Soc. Cubana Ci. Fis. Mat. 1, 115-122 (1943). (Spanish) [MF 10111]

Conkwright, N. B. An elementary proof of the Budan-Fourier theorem. Amer. Math. Monthly 50, 603-605 (1943). [MF 9850]

A very elementary proof of the Budan-Fourier theorem for the number of roots of the real equation $f(x)=0$ in the

interval $a_1 < x \leq a_2$ is given here by applying Descartes' rule of signs to the two equations $f(x+a_k)=0$ and mathematical induction and Rolle's theorem to the sequence $f'(x), f''(x), \dots, f^n(x)$. M. Marden (Milwaukee, Wis.).

Soula, J. Sur les relations qui existent entre les racines d'une équation algébrique de degré n et l'équation dérivée de degré $n-2$. Mathematica, Timisoara 19, 60-66 (1943). [MF 9957]

Let z_1, z_2, \dots, z_n be the zeros of the polynomial $f(z) = z^n + Az^{n-1} + Bz^{n-2} + \dots$ and γ_1 and γ_2 the zeros of its derivative $f^{(n-2)}(z)$ of order $n-2$. Let $\gamma = (\gamma_1 + \gamma_2)/2$; $L^2 = \sum |z_j - z_k|^2$ and $\Delta^2 = \sum \Delta_{jk}$, where Δ_{jk} is the area of the triangle with vertices z_j, z_k and γ and where in both sums $j, k = 1, 2, \dots, n$. Then, as proved by the author,

$$|\gamma_1 - \gamma_2|^2 = 4n^{-2}(n-1)^{-1}[L^2 - 16n^2\Delta^2]^{\frac{1}{2}}.$$

If all the points z_j lie in a rectangle of dimensions $a \times b$ with at least one z_j on each side of the rectangle, then

$$2n(a^2 + b^2) \leq 4L^2 \leq n^2(a^2 + b^2).$$

Thus $|\gamma_1 - \gamma_2|^2 \leq (n-1)^{-1}(a^2 + b^2)$ and for $b=0$ (for example, all z_j real)

$$2n^{-1}(n-1)^{-2}a^2 \leq |\gamma_1 - \gamma_2|^2 \leq (n-1)^{-1}a^2,$$

a theorem originally due to T. Popoviciu [Mathematica, Cluj 9, 129-145 (1935), in particular, p. 138]. In addition to the above, the author proves the theorem that the foci γ_1 and γ_2 of the ellipse tangent to the lines z_1z_2, z_1z_3 and z_2z_3 at their midpoints are the zeros of the derivative of the polynomial $(z-z_1)(z-z_2)(z-z_3)$. [Reviewer's note. This is, however, a special case of a theorem due to Van den Berg, Grace, Bôcher, Heawood, Linfield and others. For references, see M. Marden, Amer. Math. Monthly 42, 277-286 (1935).] M. Marden (Milwaukee, Wis.).

Kac, M. A correction to "On the average number of real roots of a random algebraic equation." Bull. Amer. Math. Soc. 49, 938 (1943). [MF 9695]

Cf. the same Bull. 49, 314-320 (1943); these Rev. 4, 196.

Littlewood, J. E. and Offord, A. C. On the number of real roots of a random algebraic equation. III. Rec. Math. [Mat. Sbornik] N.S. 12(54), 277-286 (1943). (English. Russian summary) [MF 10223]

[The first two papers of this series were published in J. London Math. Soc. 13, 288-295 (1938) and Proc. Cambridge Philos. Soc. 35, 133-148 (1939).] Let a_0, a_1, \dots, a_n be arbitrary but fixed complex numbers, $\lambda \geq 1$, and let $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ be arbitrarily ± 1 . It is shown that, no matter what C is, the inequality

$$|a_0 + \sum_{r=1}^n \epsilon_r a_r - C| \geq \lambda \min_{0 \leq j \leq n} |a_j|$$

holds for all the 2^n systems $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$, except for some whose total number does not exceed $2^n \cdot (2\lambda + A)n^{-1} \log n$, A an absolute constant. The exceptional systems depend on C . This result is applied to the estimation of the number of real roots possessed by "most" of the equations

$$a_0 + \epsilon_1 a_1 x + \dots + \epsilon_n a_n x^n = 0.$$

It is proved that, except for a portion of $A(\log \log n)/\log n$ of these equations, each of the remaining has a number of real roots not exceeding

$$10 \log n \{ \log (M/|a_0 a_n|)^{\frac{1}{2}} + 2(\log n)^{\frac{1}{2}} \},$$

where $M = |a_0| + |a_1| + \dots + |a_n|$. [Cf. also M. Kac, *Bull. Amer. Math. Soc.* **49**, 314-320 (1943); these *Rev.* **4**, 196.]
A. Zygmund (South Hadley, Mass.).

Zygmund, A. A property of the zeros of Legendre polynomials. *Trans. Amer. Math. Soc.* **54**, 39-56 (1943). [MF 8709]

Let $P_m(x)$ be the m th Legendre polynomial with zeros at $x_\mu^{(m)}$ and let $\gamma_\mu^{(m)}$ be the corresponding Christoffel numbers, $1 \leq \mu \leq m$. If $Q(x)$ is a polynomial of degree $n \leq m/(1+\delta)$, it is shown that

$$(1) \quad \max_{-1 \leq x \leq 1} |Q(x)| \leq A_\delta \max_{1 \leq \mu \leq m} Q(x_\mu^{(m)}),$$

$$(2) \quad \left\{ \int_{-1}^1 |Q(x)|^r dx \right\}^{1/r} \leq A_\delta \left\{ \sum_{\mu=1}^m |Q(x_\mu^{(m)})|^r \gamma_\mu^{(m)} \right\}^{1/r},$$

where A_δ is a constant which depends only on $\delta > 0$. In the proof it is shown that a polynomial can be expressed as a simple linear combination of the second Cesàro means of its Fourier-Legendre series. This leads to an inequality which is in a sense a converse of (2). Using this inequality the author proves (2). It is shown that any polynomial $R(x)$ of degree n can be written in the form $R(x) = R_1(x) - R_2(x)$, where R_1 and R_2 are of degree m , are nonnegative in the interval $(-1, 1)$ and satisfy the inequalities

$$\left\{ \int_{-1}^1 |R_1|^r \right\}^{1/r} \leq A_\delta \left\{ \int_{-1}^1 |R|^r \right\}^{1/r}.$$

There is a corresponding inequality for trigonometric polynomials. A. C. Schaeffer (Stanford University, Calif.).

Basu, K. A note on Sonine's polynomials. I. *Bull. Calcutta Math. Soc.* **35**, 21-32 (1943). [MF 9373]

Various elementary properties of Sonine's polynomials (=generalized Laguerre's polynomials) are studied.

G. Szegő (Stanford University, Calif.).

Koulik, S. Fonctions génératrices de quelques polynômes orthogonaux. *Rec. Math. [Mat. Sbornik]* **N.S. 12(54)**, 320-334 (1943). (Russian. French summary) [MF 10226]

(i) The author constructs the generating function of Kravtchouk's polynomials and discusses connections of the latter with the polynomials of Hermite and of Poisson-Charlier. This is known [cf. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939, p. 34; these *Rev.* **1**, 14]. (ii) A similar investigation is carried out for the polynomials orthogonal over $(-\infty, +\infty)$ with respect to $d\alpha(x)$, where $\alpha(x)$ has jumps

$$\binom{n}{x} \frac{a(a+\rho) \cdots (a+(x-1)\rho)b(b+\rho) \cdots (b+(n-x-1)\rho)}{(a+b)(a+b+\rho) \cdots (a+b+(n-1)\rho)}$$

at the points $x=0, 1, \dots, n$ and is constant in the adjacent intervals. Here a, b, ρ are positive integers. The distribution function $\alpha(x)$ occurs in an elementary problem of the calculus of probability. A. Zygmund.

Erdős, P. Corrections to two of my papers. *Ann. of Math.* (2) **44**, 647-651 (1943). [MF 9403]

[For the review of the second correction cf. these *Rev.* **5**, 172.] The first correction refers to the author's paper "On the divergence properties of the Lagrange interpolation polynomials" [*Ann. of Math.* (2) **42**, 309-315 (1941); cf. these *Rev.* **2**, 283]. This paper was concerned with the

Lagrange interpolation of continuous functions based on the roots of the Tchebycheff polynomials, in particular, with the exceptional role played by the set S of points $x_0 = \cos(p\pi/q)$, where $0 < p < q$, p and q odd, $(p, q) = 1$. If $x_0 \in S$, theorem 1 stated the existence of an $f(x)$ such that $L_n f(x_0) \rightarrow +\infty$. An additional assumption now required by lemma 1 replaces the conclusion of this theorem by the weaker statement that $|L_n f(x_0)| \rightarrow \infty$. On the other hand, in a note added in proof the author states that he can establish the following more general result. If $x_0 \in S$, then there exists a continuous $f(x)$ such that the set of limit points of the sequence $\{L_n f(x_0)\}$ is identical with an arbitrarily preassigned closed set E ; E may consist of the point $+\infty$ alone. I. J. Schoenberg (Aberdeen, Md.).

Grünwald, Géza. Über die Hermitesche Interpolation. *Mat. Fiz. Lapok* **48**, 272-284 (1941). (Hungarian. German summary) [MF 9225]

Let

$$\begin{array}{ccc} & x_1^{(1)} & \\ x_1^{(2)} & & x_2^{(2)} \\ & \dots & \end{array}$$

be a strongly normal point group. The author gives an elegant proof of the following result. If $f(x)$ is any continuous function then the step parabolas of $f(x)$, taken at the point group, converge uniformly to $f(x)$. [All notions mentioned here can be found in an article by Fejér, *Amer. Math. Monthly* **41**, 1-14 (1934).] P. Erdős.

Gontcharoff, W. L. Sur les facteurs correctifs de procédés d'interpolation. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **32**, 471-473 (1941). [MF 9606]

Let A denote a bounded set of points $\{a_m^{(n)}\}$, $0 \leq m \leq n$; $n=0, 1, 2, \dots$, in the complex z -plane. Denote further by $P_n(z)$ the interpolation polynomial, of degree n , such that

$$P_n(a_m^{(n)}) = f(a_m^{(n)}), \quad m=0, 1, 2, \dots, n,$$

where $f(z)$ is analytic in a certain finite domain $D \supset A$. The author's object is to determine "corrective interpolation factors" $\mu_n(z)$ such that

$$\lim_{n \rightarrow \infty} [\mu_n(z) P_n(z)] = f(z),$$

uniformly in any closed domain $D' \subset D$, the points $\{a_m^{(n)}\}$ being subject to certain conditions of "regularity." The proof, rather sketchy, is based upon the customary form of the remainder of an interpolation formula in the complex domain. Use is made of conformal mapping, and an application is indicated to Runge's case, where $a_m^{(n)} = (2m-n)/n$ and \bar{A} ($=A+A'$) is the segment of the real axis $-1 \leq x \leq 1$. J. A. Shohat (Philadelphia, Pa.).

Bernstein, S. Sur les domaines de convergence des polynômes $\sum_0^n C_n f(m/n) x^m (1-x)^{n-m}$. *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **7**, 49-88 (1943). (Russian. French summary) [MF 9580]

The paper gives a systematic discussion of the region of convergence of the Bernstein polynomials

$$B_n[f(x)] = B_n[f(z); x, 1] = \sum_{m=0}^n f(m/n) \binom{n}{m} x^m (1-x)^{n-m}$$

in the complex domain, the function $f(z)$ being analytic in a certain given region. The discussion is based upon the

following representations of $B_n[f(x)]$:

$$(1) \quad B_n[f(x)] = \frac{n!}{2\pi i} \int_C \frac{f(z) [(x-1)^{1-z} x^z] dz}{z(z-1) \cdots (z-n)}, \quad x \neq 0, 1,$$

$$(2) \quad B_n[f(x)] \sim \frac{1}{i} \left(\frac{n}{2\pi} \right)^{1/2} \int_C f(z) \left[\left(\frac{x-1}{z-1} \right)^{1-z} \left(\frac{x}{z} \right)^z \right]^n \frac{dz}{(z(1-z))^{1/2}},$$

which hold if

$$n^{-1} \int_C \left| \frac{f(z)}{(z(1-z))^{1/2}} \left[\left(\frac{x-1}{z-1} \right)^{1-z} \left(\frac{x}{z} \right)^z \right]^n \right| dz \rightarrow 0, \quad n \rightarrow \infty.$$

It is shown that the convergence of $B_n[f(x)]$ at the point x depends upon the location of the "nodal line"

$$\left| \left(\frac{x-1}{z-1} \right)^{1-z} \left(\frac{x}{z} \right)^z \right| = 1,$$

which consists in part of a loop (closed contour). A detailed exposition of this paper being out of place here, it suffices to give a few of its main results. Theorem A. If $f(z)$ is analytic in a region containing F_n and the segment 01, then the point x lies inside the region of convergence of the polynomials $B_n[f(z); x, 1] = B_n[f(x)]$. Moreover, x cannot be a point of convergence of the polynomials $B_n[f(x)]$ if $f(z)$ has one or several poles inside F_n or on its boundary. Theorem A-bis. If $f(z)$ is analytic inside F_n and on its boundary and remains finite on the segment 01, then $\lim_{n \rightarrow \infty} B_n[f(x)] = f(x)$. The paper closes with the study of the region of convergence of the polynomials

$$A_n[f(x)] = \lim_{h \rightarrow 0} B_n[f(z); x, h],$$

where

$$B_n[f(z); x, h] = B_n[f(hz); x/h, 1] \\ = \sum_{m=0}^n f(hm/n) \binom{n}{m} (x/h)^m (1-x/h)^{n-m}.$$

Application is made to deriving a criterion of analyticity for the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ at a certain point x_0 on the circumference of its circle of convergence.

J. A. Shohat (Philadelphia, Pa.).

Korovkin, P. Sur la divergence des séries de polynômes. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 179-181 (1941). [MF 9626]

The results of a previous note [C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 531-534 (1940); these Rev. 2, 194] are extended. Let C be a closed analytic curve and let r be the transfinite diameter of C . Let $\{p_n(x) = x^n + \dots\}$ be a sequence of normalized polynomials, let $\lim_{n \rightarrow \infty} |c_n|^{1/n} = 1/r$ and let the series $\sum c_n p_n(x)$ be uniformly convergent on C . Then it diverges almost everywhere in the exterior of C . G. Szegő (Stanford University, Calif.).

Special Functions

Fock, V. A. On the representation of an arbitrary function by an integral involving Legendre's functions with a complex index. C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 253-256 (1943). [MF 9829]

If Laplace's equation is expressed in toroidal coordinates ϑ, φ , where

$$r = \sqrt{x^2 + y^2} = \frac{\sinh \vartheta}{\cosh \vartheta - \cos \varphi}, \quad z = \frac{\sin \varphi}{\cosh \vartheta - \cos \varphi},$$

a solution can be obtained in terms of the Legendre function $P_{\mu-1}(\cosh \vartheta)$, where $\mu^2 > 0$. Boundary conditions may then require the inversion of the integral

$$\Psi(x) = \int_0^{\infty} P_{\mu-1}(x) f(\mu) d\mu, \quad 1 \leq x < \infty.$$

The author gives a formal inversion theorem

$$f(\mu) = \mu \tanh \mu \pi \int_1^{\infty} P_{\mu-1}(x) \Psi(x) dx,$$

and states the conditions under which it is valid.

M. C. Gray (New York, N. Y.).

Bateman, H. Note on the function $F(a, b; c-n; z)$. Proc. Nat. Acad. Sci. U. S. A. 30, 28-30 (1944). [MF 10113]

For the function given in the title asymptotic expressions are obtained as $n \rightarrow \infty$. Darboux's method is used.

G. Szegő (Stanford University, Calif.).

Mohan, B. A certain confluent hyper-geometric function. Proc. Nat. Acad. Sci. India 11, 78-83 (1941). [MF 9677]

This paper contains essentially the same results as the one reviewed below. M. A. Basoco (Lincoln, Neb.).

Mohan, B. A confluent hyper-geometric function. Proc. Nat. Inst. Sci. India 7, 177-182 (1941). [MF 9675]

In a former paper [Proc. Edinburgh Math. Soc. (2) 4, 53-56 (1934)] the author proved that, if $f(x)$ is self-reciprocal for J_1 transforms, the function

$$(1) \quad g(x) = x^{-\nu/2} \int_0^{\infty} y^{-\nu/2} H_{\nu-1}(xy) f(y) dy,$$

where $H_{\nu}(x)$ is Struve's function of order ν , is self-reciprocal for J_{ν} transforms. It is also known that $x^{\nu+1} e^{-x^2/2}$ is self-reciprocal for J_{ν} transforms. Hence setting $f(x) = x^{\nu} e^{-x^2/2}$ in (1), it may be shown that

$$g(x) = 2^{-\nu/2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^m \Gamma(\nu/2 + m + 1)},$$

so that the function

$$\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^m \Gamma(\nu/2 + m + 1)}$$

is self-reciprocal for J_{ν} transforms. This leads the author to an examination of the function

$$(2) \quad f_r(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{r+2m}}{2^{r+2m} \Gamma(\nu + r + 1)}.$$

The function $x^{1-\nu/2} f_{\nu/2}(x\sqrt{2})$ is self-reciprocal for J_{ν} transforms. It is clear that

$$f_0(x) = e^{-x^2/2}; \quad f_{\nu}(x) = ((x/2)^{\nu} / \Gamma(\nu+1)) {}_1F_1(1; \nu+1; -x^2/4).$$

The object of this paper is to study the properties of this function. A great many relations and identities are given which are too numerous to list here. M. A. Basoco.

Mohan, B. Properties of a confluent hyper-geometric function. Proc. Nat. Inst. Sci. India 8, 93-97 (1942). [MF 9674]

This paper is a continuation of the one reviewed above. The function studied is

$$g_r(x) = \sum_{m=0}^{\infty} \frac{(x^{r+2m} / 2^{r+2m} \Gamma(\nu + r + 1))}{\Gamma(\nu + r + 1)}$$

[compare with equation (2) above]. The properties of this function are quite similar to that defined above. [See also these Rev. 4, 82.] *M. A. Basoco* (Lincoln, Neb.).

Mohan, B. A class of infinite integrals. I. Science and Culture 7, 362-364 (1941). [MF 9679]

This paper is concerned with the evaluation of the integral

$$I = \int_0^\infty x^{-p} e^{-x^2/2} J_m(bx) g_{m+p+1}(cx) dx,$$

where $g(x)$ is as defined in the preceding review. It is shown that

$$I = \frac{b^{p+m+1} e^{-b^2/2c^2}}{2^p c^m \Gamma(p+m+1)} W_{-p/2, -1}(b^2/c^2),$$

where $W_{k,m}(z)$ is Whittaker's function. Numerous special cases of this result are stated. *M. A. Basoco*.

Mohan, B. Infinite integrals involving Struve's functions.

II. Proc. Nat. Acad. Sci. India 12, 231-235 (1942). [MF 9676]

[The first part appeared in Quart. J. Math., Oxford Ser. 13, 40-47 (1942), the third part in Bull. Calcutta Math. Soc. 34, 55-59 (1942); cf. these Rev. 4, 82, 141.] The main result given is as follows. If

$$I = \int_0^\infty x^{l-1} e^{-x^2/2} W_{k,m}(x^2) H_r(bx) dx,$$

where $W_{k,m}(z)$ is Whittaker's function and $H_r(z)$ is Struve's function [see G. N. Watson, Theory of Bessel Functions, Cambridge, 1922, §10.4], then

$$I = b^{l+1} \Gamma\left(\frac{l}{2} + \frac{p}{2} + m + 1\right) \Gamma\left(\frac{l}{2} + \frac{p}{2} - m + 1\right) \times {}_2F_3\left(\begin{matrix} 1, (l+p)/2 + m + 1, (l+p)/2 - m + 1, \\ \frac{3}{2}, \frac{p}{2} + \frac{3}{2}, (l+p)/2 - k + \frac{3}{2}, \end{matrix}; -\frac{b^2}{4}\right),$$

where $\Re(p) > -1$, $\Re(l+p \pm 2m) > -2$. Numerous particular cases are listed. *M. A. Basoco* (Lincoln, Neb.).

Shanker, Hari. On confluent hypergeometric functions which are Hankel-transforms of each other. J. Indian Math. Soc. (N.S.) 7, 63-67 (1943). [MF 10101]

Write

$$\varphi(n, m; x) = 2^n \Gamma(2n + \nu + 1) e^{1/2 x^2} x^{2m-1} W_{k,m}(\frac{1}{2} x^2),$$

$k = -(\nu + 2n + m + \frac{1}{2})$. Then $\varphi(n, m; x)$ and $\varphi(m, n; x)$ are Hankel transforms of each other (of order ν) if $\Re(\nu+1)$, $\Re(\nu+2m+1)$, $\Re(\nu+2n+1)$, $\Re(\nu+4m+\frac{1}{2})$, $\Re(\nu+4n+\frac{1}{2})$ are all positive. This result was previously obtained, with a shorter derivation, by W. N. Bailey [J. London Math. Soc. 5, 258-265 (1930)]. *R. P. Boas, Jr.* (Cambridge, Mass.).

Buchholz, Herbert. Die konfluente hypergeometrische Funktion mit besonderer Berücksichtigung ihrer Bedeutung für die Integration der Wellengleichung in den Koordinaten eines Rotationsparaboloides. Z. Angew. Math. Mech. 23, 47-58, 101-118 (1943). [MF 9784]

This is an expository paper summarizing the properties of the confluent hypergeometric functions $M_{k,m}(z)$ and $W_{k,m}(z)$. The author is particularly interested in the functions for which the parameter k and the argument z are purely imaginary and shows that these functions occur as solutions of the wave equation in paraboloidal coordinates.

Tables of values of the function

$$m_{ir}(i\zeta) = [\Pi/2i\zeta]^{\frac{1}{2}} M_{ir,0}(i\zeta)$$

and its first derivatives are given, and the smallest zeros of m_{ir} and m_{ir}' for positive integral values of ζ are also tabulated. *M. C. Gray* (New York, N. Y.).

Argence, E. Sur une dégénérescence des fonctions d'Appell. C. R. Acad. Sci. Paris 213, 817-820 (1941). [MF 9657]

The functions mentioned in the title are

$$\tilde{\omega}_k = n^{-1} \sum_{r=0}^{n-1} j_{k-1}^r \exp \{j_r \phi_1 + j_r^2 \phi_2 + \dots + j_r^{p-1} \phi_p\},$$

$$k=1, 2, \dots, n,$$

where $p < n$ and $j_0 = 1, j_1, \dots, j_{n-1}$ are the n th roots of unity. The partial differential equation satisfied by the n functions

$$U_k = e^{-(\lambda_1 x_1 + \dots + \lambda_p x_p)} \tilde{\omega}_k(\lambda_2 x_2, \lambda_3 x_3, \dots, \lambda_p x_p), \quad k=1, 2, \dots, n,$$

as functions of the x_i (the λ_i being arbitrary constants) is written down in the form of a determinant and the connection of this differential equation with $(n-p)$ -dimensional manifolds in an n -dimensional Borel space [same C. R. 195, 992-994 (1932)] is pointed out. *A. Erdélyi* (Edinburgh).

Therrien, K. A. Numerische Untersuchungen über die Emden-Funktionen. I. Z. Astrophys. 22, 122-155 (1943). [MF 9906]

The Lane-Emden function $\theta_n(\xi)$ of index n is the solution of

$$\frac{d^2 \theta}{d\xi^2} + \frac{2}{\xi} \frac{d\theta}{d\xi} + \theta^n = 0,$$

n a real constant, which satisfies the boundary conditions $\theta=1$ and $d\theta/d\xi=0$ at $\xi=0$. These functions, particularly in their dependence on the index n , are studied from a novel point of view. Expressing θ_n as a power series of the form

$$\theta_n = \sum_{r=0}^{\infty} a_r(n) \xi^{2r}, \quad a_0(n) = 1,$$

convergent for ξ less than a certain value (depending on n), the author observes that the ratio $f_r(n) = a_r(n)/a_{r-1}(n)$ of the successive coefficients rapidly converges to a limiting value $f_{lim}(n)$ as $r \rightarrow \infty$. This convergence is particularly rapid for $n=3$, in which case $f_3(3)$ already does not differ from the limiting value by more than one part in a thousand. Accordingly the author investigates whether $f_r(n)$ can be expanded as a Laurent series in the form

$$f_r(n) \sim f_{lim}(n) \left\{ 1 + \sum_{j=1}^{\infty} (\epsilon_j(n)/\nu^j) \right\},$$

and by a combination of analytical and empirical reasoning deduces for $f_r(n)$ the representation

$$f_r(n) \sim f_{lim}(n) \left(1 - \frac{n-3}{n-1} \frac{1}{\nu} - \frac{\alpha n-3}{8 n-1} \frac{n-5}{n-1} \frac{1}{2\nu(2\nu+1)} \right),$$

where

$$\alpha = \lim_{n \rightarrow 1} ((n-1)^2/f_{lim}(n)).$$

Now a power series $\sum c_r \xi^{2r}$ with

$$\frac{c_r}{c_{r-1}} = f_{lim}(n) \left(1 - \frac{n-3}{n-1} \frac{1}{\nu} \right), \quad c_0 = 1,$$

can be summed to

$$U_n(\xi) = [1 - f_{lim}(n) \xi^2]^{-2/n-1}.$$

A comparison of $\theta_n(\xi)$ with $U_n(\xi)$ shows that the latter provides a surprisingly good representation of the former for $3 \leq n \leq 5$ over most of the range of ξ that is of astrophysical interest. The values of $f_{lim}(n)$ for a number of values of n are numerically evaluated with the aid of a recursion formula for $f_n(n)$.

In view of the fair representation of θ_n provided by $U_n(\xi)$, the author next investigates an expansion for θ_n of the form

$$\theta_n(\xi) = 1 + \sum_{k=1}^{\infty} b_k [1 - U_{n,k}(\xi)]^r,$$

where

$$U_{n,k}(\xi) = (1 - k\xi^2)^{-2/n-1}, \quad k \text{ a constant,}$$

and shows that such an expansion convergent for all ξ can in fact be found. However the convergence appears to be most rapid when k is chosen to be equal to $f_{lim}(n)$. It is finally demonstrated that a simple formula of the type

$$U_n(\xi) = 1 - \frac{n-1}{12f_{lim}(n)} [1 - U_n(\xi)] + b_1(n) \frac{1 - U_n(\xi)}{U_n(\xi)}$$

with a suitable choice of $b_1(n)$ provides a representation of θ_n in the range $0 \leq \xi \leq \xi_1$ (where ξ_1 denotes the first zero of θ_n) which is correct to within a fraction of a percent. A table of the coefficient $b_1(n)$ in the foregoing formula is also provided. *S. Chandrasekhar* (Williams Bay, Wis.).

Differential Equations

Lowry, H. V. An operational method of solving linear differential equations. *Math. Gaz.* 26, 161-164 (1942). [MF 10129]

Under the transformation $x = e^t$, $xd/dx = d/dt = \Delta$, Bessel's equation

$$(1) \quad x^2 d^2 y / dx^2 + x dy / dx + (x^2 - n^2) y = 0$$

takes the form $(\Delta^2 + n^2 + e^{2t})y = 0$. Define the sequence $\{u_i\}$ by the equations $(\Delta^2 - n^2)u_1 = 0$, $(\Delta^2 - n^2)u_{i+1} = -e^{2t}u_i$, ($i = 1, \dots$). The author solves for each u_i by operational methods and then finds a solution of the equation of (1) in the form $y = \sum u_i$. *F. G. Dressel* (Durham, N. C.).

Levinson, Norman. On a non-linear differential equation of the second order. *J. Math. Phys. Mass. Inst. Tech.* 22, 181-187 (1943). [MF 9776]

The differential equation is

$$(*) \quad \ddot{x} + f(x) \cdot \dot{x} + x = e(t).$$

The function $e(t)$ is assumed to be continuous and periodic of least period L . The function $f(x)$ is assumed to be piecewise continuous. The object of the paper is to show, in the terminology of mechanics, that every motion will tend to a unique periodic motion of least period L if the "damping coefficient" $f(x)$ is nonnegative, that is, if the force depending on the velocity is truly a resisting force. More precisely, the following theorem is proved. If $f(x) > 0$, except possibly at discrete points, and if $\int_{-\infty}^{\infty} f(x) dx = \infty$ (or if $\int_{-\infty}^{\infty} f(x) dx = \infty$), then the differential equation (*) possesses a periodic solution of period L toward which all other solutions tend as $t \rightarrow +\infty$. The difficult part of the theorem to prove is that concerning the existence of a periodic solution. Once this is assured, the remainder of the theorem is proved rather

easily. The existence proof follows much the same lines as that used by the author for a different case (On the existence of periodic solutions for second order differential equations with a forcing term [*J. Math. Phys. Mass. Inst. Tech.* 22, 41-48 (1943); these Rev. 5, 66]).

J. J. Stoker (New York, N. Y.).

Pierce, Jesse. Solutions of systems of differential equations in the vicinity of branch points of the solutions. II. *Duke Math. J.* 11, 83-88 (1944). [MF 10149]

The paper is a generalization of one by this author on a single equation [*Duke Math. J.* 4, 650-655 (1938)]. It deals with the system of differential equations

$$x_i \frac{dx_i}{dt} = (m+1)^{-1} + f_i(t) + \sum_{j=1}^m f_{ij}^{(v)}(t) x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n},$$

$$i = 1, 2, \dots, n,$$

in which m is a positive integer, $\mu_1 + \mu_2 + \dots + \mu_n = v$, and μ represents $(\mu_1, \mu_2, \dots, \mu_n)$. The coefficients $f_i(t)$, $f_{ij}^{(v)}(t)$ are assumed to be uniformly bounded and to be integrable on the straight segment (t_0, t) . It is shown that, if these coefficients are analytic at t_0 , and if at least one of the functions $\{(m+1)^{-1} + f_i(t)\}$ is not identically zero, there exist at least $(m+1)$, and in general $(m+1)^n$, distinct solutions with $x_i(t_0) = 0$. The method utilizes a substitution $x_i = \sum_{j=1}^n y_j y_{ij}$, $i = 1, 2, \dots, n$, in which the y_{ij} are successively determinable, and employs dominant series to establish convergence. *R. E. Langer* (Madison, Wis.).

Kamke, E. Bemerkungen zur Theorie der partiellen Differentialgleichungen erster Ordnung. *Math. Z.* 49, 256-284 (1943). [MF 10037]

The paper deals with the system of partial differential equations

$$(1) \quad \partial z / \partial x_r = f^{(r)}(x_1, \dots, x_r, y_1, \dots, y_s, z, \partial z / \partial y_1, \dots, \partial z / \partial y_s, \lambda_1, \dots, \lambda_m), \quad r = 1, 2, \dots, r,$$

and with the single equation here represented when $r = 1$. When $r > 1$ the system is assumed to fulfill the conditions of integrability by virtue of which it is involutory. The variables and the parameters $\lambda_1, \lambda_2, \dots, \lambda_m$ are real. The matter at issue is the specification of a region

$$|x_r - \xi_r| \leq \alpha$$

about a given point $(\xi_1, \xi_2, \dots, \xi_r)$, in which there exists a unique integral

$$z = \psi(x_1, \dots, x_r, y_1, \dots, y_s, \lambda_1, \dots, \lambda_m),$$

possessed of continuous partial derivatives of prescribed orders and such that

$$z|_{x_r = \xi_r} = \omega(y_1, \dots, y_s, \lambda_1, \dots, \lambda_m),$$

with a given function ω . The hypotheses and the relations through which α is determined in terms of bounds upon the derivatives of the functions $f^{(r)}$ and ω are too lengthy to be stated here. The theorem is separately stated for the case in which the system is linear in z and its partial derivatives, as it also is for $r = 1$ and $r > 1$. *R. E. Langer*.

Oberhettinger, F. Über ein Randwertproblem der Wellengleichung in Zylinderkoordinaten. *Ann. Physik* (5) 43, 136-160 (1943). [MF 9930]

Let C be the surface of an infinite cylinder of radius b . Let D be a harmonic dipole at distance a from the cylinder axis; the axis of D is parallel to the axis of the cylinder.

The object of the paper is the computation of the electromagnetic field if C is a perfect conductor. If $Ae^{-i\omega t}$ is the vector potential of this field, A may be written as $A_1 + A_2$, where A_1 corresponds to the primary field, that is, the field of the dipole without the presence of the conductor C . After putting the known potential A_1 into a suitable form the author sets out to compute A_2 , which leads to the following boundary value problem: $A_2 = -A_1$ on C and $\Delta A_2 + k^2 A_2 = 0$, where the differential equation has to be satisfied inside C in the interior case ($a < b$) and outside C in the exterior case ($a > b$). In both cases an integral representation for the solution is given. For the interior case, moreover, the author succeeds in giving the solution in the form of an expansion containing trigonometric and cylinder functions. In the exterior case an approximation for the field at large distance from the dipole is given which then is shown to be the first term of a general asymptotic expansion for large distance. The paper is concluded by some numerical calculations for large distance. *E. H. Rothe.*

Sircar, H. On harmonic function. *Bull. Calcutta Math. Soc.* **35**, 71-75 (1943). [MF 9952]

With the aid of a result found by Villat in 1916 it is shown that, if $i\pi S = \omega_1 \log w - i\omega_2 s$, $i\pi S' = \omega_1 \log w + i\omega_2 s$, $q = i\pi\omega_2/\omega_1$, $w = u + iv$, $f(2\pi - s) = f(s)$, where $\log w$ has its chief value and $\omega_1, 2\omega_2$ are the semi-periods of the Weierstrassian functions ζ, ξ , then the function

$$E(w) = (i\omega_1/\pi^2) \int_0^{2\pi} f(s) ds [\zeta(S) + \zeta(S') - \zeta_2(S) - \zeta_2(S')]$$

is such that $E(w) - f(\theta)$ is imaginary at each point $w = \exp(i\theta)$ on the semicircle $|w| = 1$, $v > 0$; $E(w)$ is real on the lines $v = 0$, $q < u < 1$, $-1 < u < -q$ while its normal derivative is zero on the semicircle $|w| = q$, $v > 0$. Studies are then made of various conformal transformations some of which depend upon the Weierstrassian function \wp with periods $2\omega_1' = -2i\omega_1$, $2\omega_2' = 2i\omega_2$. The analysis is preliminary to a treatment of the transverse oscillations in a long canal of circular section. *H. Bateman* (Pasadena, Calif.).

Copson, E. T. On Whittaker's solution of Laplace's equation. *Proc. Roy. Soc. Edinburgh. Sect. A.* **62**, 31-36 (1944). [MF 10120]
Whittaker's solution

$$V = (2\pi)^{-1} \int_0^{2\pi} f(z + ix \cos u + iy \sin u, u) du$$

of Laplace's equation is shown to hold under the following conditions. If $V(x, y, z)$ is a solution of Laplace's equation in the sphere $x^2 + y^2 + z^2 < r_0^2$, $U = z + ix \cos u + iy \sin u$ and $F_n(u)$ is a trigonometric polynomial of degree n at most, then V can be represented in the form

$$V = (2\pi)^{-1} \int_0^{2\pi} \sum_0^n F_n(u) U^n du,$$

for $x^2 + y^2 + z^2 < \frac{1}{4}r_0^2$. The following representation of the fundamental solution of Laplace's equation is also obtained:

$$\frac{1}{((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{3/2}} = \frac{1}{4\pi^2} \int_0^{2\pi} du \int_0^{2\pi} dt \frac{T_0 + U}{T_0^2 - 2T_0U \cos(t-u) + U^2}$$

where $T_0 = z_0 + ix_0 \cos t + iy_0 \sin t$.

F. G. Dressler.

Parodi, Hippolyte. Sur une solution particulière des équations de l'élasticité. *C. R. Acad. Sci. Paris* **216**, 172-173 (1943). [MF 10010]

By use of the harmonic function $f(ax + \beta y + \gamma z)$, $\alpha^2 + \beta^2 + \gamma^2 = 0$, and the biharmonic function $F = (lx + my + nz)f$, the author constructs particular solutions of the equations of elasticity. In particular, expressions for the displacements are determined in terms of the arbitrary harmonic function f and five arbitrary parameters, which can be used to satisfy boundary conditions. It is pointed out that the principle of superposition permits an indefinite increase in the number of arbitrary harmonic functions and arbitrary parameters. *G. E. Hay* (Providence, R. I.).

Schmidl, Oskar. Die Potentialverteilung in einem unendlich langen leitenden Hohlkegel, hervorgerufen durch symmetrisch zur Kegelschse angeordnete elektrische Ladungsbereiche. *Ann. Physik* (5) **43**, 193-202 (1943). [MF 9925]

Let C be the surface of an infinite cone and u_0 the potential of given electric charges which are distributed symmetrically with respect to the axis of the cone in the interior of C . Assuming C to be a perfect conductor, the author computes the potential u of the electric field inside C . Since $u = u_0 + u_1$, where u_1 is the potential of the charges induced on C by the given charges, the main problem consists in the computation of u_1 . This leads to the following boundary value problem: $\Delta u_1 = 0$ inside C and $u_1 = -u_0$ on C . The case where the given charges are distributed in the form of an infinitesimal ring is treated by separating the variables and using an integral representation analogous to the Fourier integral; thus one obtains u_1 in the form of a triple integral which is then changed to a complex line integral which, in turn, is evaluated as a sum of residues. The solution is explicitly obtained as a series proceeding according to the zeros of Legendre polynomials. The case of a general symmetric distribution can then be obtained by adding the potentials of ring distributions. The case of a cone degenerating into a plane is also treated.

E. H. Rothe (Ann Arbor, Mich.).

Tikhonov, A. N. On the stability of inverse problems. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **39**, 176-179 (1943). [MF 10045]

An inverse problem such as that of determining what uniform distribution of mass below a given surface will produce a known potential on that surface has, if only suitable restricted mass distributions are considered, a unique solution. Approximation methods can be applied to its calculation provided it can be proved that the general mass distribution is a continuous function of its potential on the surface, that is, provided the solution of the inverse problem is stable. The author shows that stability in the above case is a consequence of the topological theorem that a continuous function with a compact domain has a continuous inverse at any point where the inverse is single-valued. As a second application of this theorem to inverse problems the author demonstrates the continuous dependence of solutions of ordinary differential equations upon a parameter. *L. H. Loomis* (Cambridge, Mass.).

Bateman, H. The transformation of partial differential equations. *Quart. Appl. Math.* **1**, 281-296 (1944). [MF 9907]

This paper is a good survey of some of the numerous studies which have been made of transformations of partial

differential equations. These transformations are classified into two major groups: (A) those arising from the conditions or conditions that a linear differential form be of a specified type (for example, an exact differential) and (B) those arising from the conditions that a quadratic differential form be of a specified type (for example, the transformation of a linear differential equation to a form in which the variables are separated). The paper is too detailed to abstract in full. An excellent bibliography is appended.

A. E. Heins (Cambridge, Mass.).

Special Functional Equations

Ionescu, D. V. Quelques applications de certaines équations fonctionnelles. *Mathematica, Timișoara* 19, 159-166 (1943). [MF 9963]

Alaci, V. Sur deux équations fonctionnelles. *Mathematica, Timișoara* 19, 23-25 (1943). [MF 9956]

This note contains the theorem that all solutions of the functional equations

$$f(x+y) = \frac{f(x)+f(y)-2f(x)f(y)}{1-2f(x)f(y)},$$

$$g(x+y) = \frac{g(x)+g(y)-1}{2g(x)+2g(y)-2g(x)g(y)-1}$$

are, respectively, of the forms

$$f(x) = \frac{1}{1+\cot ax}, \quad g(x) = \frac{1}{1+\tan bx},$$

with constant a, b . The author's proof consists in reducing those equations to the ones treated by Anghelutza [*Mathematica, Timișoara* 19, 19-22 (1943); these Rev. 5, 72]. However, this reduction is carried out in a manner not sufficiently justified.

F. John (Aberdeen, Md.).

Fan, Ky. Une propriété asymptotique des solutions de certaines équations linéaires aux différences finies. *C. R. Acad. Sci. Paris* 216, 169-171 (1943). [MF 10009]

The note gives a proof of the following theorem. "Given the linear difference equation

$$\Delta^m y(n) + g_{m-1}(n) \Delta^{m-1} y(n) + \dots + g_1(n) \Delta y(n) + g_0(n) y(n) = h(n).$$

If the series

$$\sum_{n=1}^{\infty} n^{m-k-1} |g_k(n)|, \quad k=0, 1, \dots, (m-1),$$

$$\sum_{n=1}^{\infty} h(n)$$

are convergent, then for any solution $y(n)$ the limits

$$\lim_{n \rightarrow \infty} \frac{(m-k-1)!}{n^{m-k-1}} \Delta^k y(n), \quad k=0, 1, \dots, (m-1),$$

exist and are equal." The theorem and its proof are analo-

gous to ones given by O. Haupt for differential equations [*Math. Z.* 48, 289-292 (1942); these Rev. 4, 276].

R. E. Langer (Madison, Wis.).

Gerceanoff, N. Quelques procédés de la résolution des équations fonctionnelles linéaires par la méthode d'itération. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 39, 207-209 (1943). [MF 10039]

The author extends earlier work on linear functional equations [*C. R. (Doklady) Acad. Sci. URSS (N.S.)* 31, 835-836 (1941); these Rev. 3, 298]. First the linear equation

$$(1) \quad Af(\varphi) + f(x) = \Phi$$

is discussed (A, φ, Φ being given functions of x , and f to be determined), with the condition that $\varphi(x)$ be a reflexive function, that is, $\varphi(\varphi(x)) = x$. The case where $A(x)$ is arbitrary is shown to be reducible to the simpler case where $A(x)A(\varphi) = 1$; and in this situation a solution of (1) is offered in the parametric form

$$(2) \quad f = (n/2)[A\Phi_1 - \Phi] + \Phi t + A\Phi_1(\frac{1}{2} - t),$$

with (a) $x = \varphi_n(x_0)$. Here n is the parameter (of which more will be said later), x_0 is an arbitrary constant, t is an arbitrary function of x and subscripts indicate iterations (that is, replacement of x by φ). If $x = \varphi_n(x_0)$ is iterated, n is replaced by $n+1$, and in showing that (2) yields a solution of (1), one iterates (2), replacing n by $n+1$. The reviewer is of the opinion that herein lies a gap in the argument. In effect, in order that (2) be a solution of (1), it is necessary that n be a function of x : $n = N(x)$, with the property that $N(\varphi) = N(x) + 1$. This is itself an equation of type (1), for which the author believes he obtains a solution by solving the relation (a) for n in terms of x . However, no evidence is offered (save a particular example or two) that (a) can be solved for n in terms of x . Apart from this point, the methods used are interesting and are shown to extend to more general linear equations.

I. M. Sheffer.

Hadamard, J. Two works on iteration and related questions. *Bull. Amer. Math. Soc.* 50, 67-75 (1944). [MF 9888]

The problem of iteration consists in finding a one-parameter family of functions $f(n, x)$ corresponding to a given function $f(x)$ such that $f(1, x) = f(x)$ and $f(m, f(n, x)) = f(m+n, x)$. Save for special cases, the problem has been solved only in a formal sense for power series $f(x) = \sum_{k=1}^{\infty} a_k x^k$, which need not converge but for which a_1 is not a root of unity. Three methods have been proposed, all of which lead to essentially the same formal results. The first, proposed by G. D. Birkhoff in a slightly different problem and later developed by the reviewer in a form explicitly applicable to the present problem [*Duke Math. J.* 5, 794-805 (1939)], is based on the use of difference equations. The second method, proposed by Jabotinsky, is based on the partial differential equation

$$\frac{\partial^2 f}{\partial n^2} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x \partial n} \frac{\partial f}{\partial n}$$

The third method, proposed by Lüntz, is based on a theory of "reversible functions," two functions $f(x)$ and $g(x)$ being "reversible" if $f(g(x)) = g(f(x))$. The author publishes the last two of these methods in behalf of their discoverers and takes the opportunity of adding a number of interesting comments which touch upon all three methods.

D. C. Lewis (New York, N. Y.).

Functional Analysis, Ergodic Theory

Wassilkoff, D. Classification of orderings of linear systems. C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 167-169 (1943). [MF 10043]

Let $M = \{x, y, \dots\}$ be an arbitrary set and let Ω denote any law by which a partial ordering (hereafter, simply "ordering") is attached to the elements of M . To distinguish between different orderings we write $x < y[\Omega]$. If Ω' and Ω'' are two orderings of M , we shall say Ω' is included in Ω'' and write $\Omega' \subseteq \Omega''$ if $x < y[\Omega']$ implies $x < y[\Omega'']$. Thus, the totality $\overline{\mathcal{M}}$ of all orderings of M is itself an ordered set. The "void" ordering is the ordering in which any two elements of M are incomparable, and it may be considered as being included in any ordering. If $\{\Omega^*\}$ is any set in $\overline{\mathcal{M}}$, then $\overline{\mathcal{M}}$ contains an ordering Ω called the "meet" of the system $\{\Omega^*\}$ and defined as follows: $x < y[\Omega]$ if $x < y[\Omega^*]$ for all Ω^* . A "join" may also be defined for systems $\{\Omega^*\}$ bounded above; Ω is said to be "irreducible" if it is not the meet of orderings in $\overline{\mathcal{M}}$ different from Ω and to be "strict" if any two elements of M are comparable $[\Omega]$. Theorem: an ordering $\Omega \in \overline{\mathcal{M}}$ is irreducible if and only if it is strict. Any $\Omega \in \overline{\mathcal{M}}$ can be represented (not generally uniquely) as the meet of a system of strict orderings whose cardinal number does not exceed that of M .

The author next considers orderings of a linear system E (an Abelian group with real numbers as operators), subjecting them to the axioms: (I) if $x < y$, then $x + z < y + z$ for any $z \in E$; (II) if $x < y$ and $\lambda > 0$, then $\lambda x < \lambda y$. The set \mathcal{E} of all such "linear orderings" of E is a closed subset of $\overline{\mathcal{E}}$, that is, \mathcal{E} contains the least upper bound (greatest lower bound) of any subset of \mathcal{E} bounded above (below) in $\overline{\mathcal{E}}$. A theorem on reducibility of linear orderings analogous to the one stated above is given. The author introduces various axioms concerning the set $R_x = E[\lambda x, 0 \leq \lambda < \infty]$. R (regularity): if R_x is bounded above but not below, then $x < 0$. H^* (weak homogeneity): for any $x > 0$, R_x is not bounded above. H (homogeneity): if R_x is bounded above and below, then $x = 0$. The interplay between these axioms is studied. Theorem: among all regular orderings including $\Omega \in \mathcal{E}$ there exists a minimal ordering Ω' , "the regular hull over Ω ," included in any regular ordering including Ω . Theorem: the linear ordering Ω is homogeneous if and only if there exists an ordering $\Omega' \supseteq \Omega$, where Ω' satisfies R and H . The author finally considers relations between orderings in E and orderings in the adjoint space E' . These results are connected with those in an earlier paper by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) 35, 135-137 (1942); these Rev. 4, 162]. No proofs are given.

J. V. Wehausen (Columbia, Mo.).

Wassilkoff, Dimitry. On the theory of partially ordered linear systems and linear spaces. Ann. of Math. (2) 44, 580-609 (1943). [MF 9399]

This paper treats (partially) ordered linear systems, that is, (partially) ordered Abelian groups admitting real multiplication. The results had been announced earlier without proof [C. R. (Doklady) Acad. Sci. URSS (N.S.) 35, 135-137 (1942); these Rev. 4, 163]. They bear especially on the rôle played in various extension, imbedding and representation theorems by analogues of the familiar Archimedean property so important for linearly ordered systems. A careful investigation convinces the reviewer that several of the theorems presented here are false. The main difficulty is

encountered in theorem 2.4, on which a good part of the further development is based. A simple counter-example is afforded by the Archimedean system E consisting of all everywhere-defined Lebesgue-measurable functions on the unit interval in their natural ordering, the subsystem E' consisting of the bounded functions in E , and the positive linear function p' coinciding with the Lebesgue integral for all numbers of E' : the latter function cannot be extended to be positive linear over E since some positive functions in E necessarily have "infinite integrals"! The author's proof seems to contain two flaws, each of which tends to conceal an unfavorable aspect of the possible relations among E , E' and p' . The first occurs in the assertion that " Φ is closed in Σ , consequently Φ is bicomact in itself," an assertion which is countered by the example where E is the real number system and Φ , being the group of characters of the additive group E , is known to have the same topological structure as E itself. The second occurs in the assertion that "there exists a positive linear function $p_1(x)$ on E_1 that coincides with $p'(x)$ on E_2 ." This assertion is identical with theorem 2.4 in the special case where E has a finite linear basis, and can be countered by the following example. Let E be the system of vectors v with three real components $x, y, z, v = (x, y, z)$, where $v_1 < v_2$ if and only if (a) $0 < x_2 - x_1, 0 < y_2 - y_1$ or (b) $0 < x_2 - x_1, 0 = y_2 - y_1, 0 < z_2 - z_1$ or (c) $0 = x_2 - x_1, 0 < y_2 - y_1, 0 < z_2 - z_1$. Now E is a "weakly homogeneous" system containing as a proper subsystem the set E' of all vectors $v' = (0, y, z)$. For the latter vectors we define $p'(v') = z$, so that p' is positive linear. It results that p' cannot be extended over E since $x = y > 0$ implies $\lambda(0, 0, z) < (x, y, 0)$ for all λ . The failure of theorem 2.4 imperils theorems 2.5, 3.1, 4.1, 4.2, 4.3, 6.2 and 7.2, as well as lemmas 2.3 and 4.1. Lemma 4.1 can be proved by an independent argument; and theorem 4.2 is certainly true for some suitably defined ordering of E , though the reviewer's studies do not settle the ordering actually introduced. Except for theorems 4.3, 6.2 and 7.2, which have not been fully examined, the remaining numbered propositions can all be countered by appropriate examples. For instance, the representation ascribed to any Archimedean system by theorem 3.1 cannot exist in the case of the system of all real functions f defined and continuous at every point of the unit interval outside a finite set A_f varying with f , the order relation $f_1 \leq f_2$ signifying that $f_1(x) \leq f_2(x)$ except possibly for a finite set of x 's. The results of the paper in §§5-7 (except for theorems 6.2 and 7.2) do not depend on theorem 2.4. In §5 it is shown that every Archimedean system can be completed by means of a generalized Dedekind-cut construction, a result well-known in the analogous case of ordered Abelian groups [see Clifford, Ann. of Math. (2) 41, 465-473 (1940); these Rev. 2, 4]. In §§6-7 a few simple connections between ordering and norm-topology are derived. It may be pointed out that valid representation theorems for Archimedean systems are obtainable by combining the results of §5 with those announced by the reviewer for completely ordered Abelian groups [Proc. Nat. Acad. Sci. U. S. A. 27, 83-87 (1941); these Rev. 2, 318].

M. H. Stone (Washington, D. C.).

Julia, Gaston. Sur la convergence faible. C. R. Acad. Sci. Paris 216, 97-100 (1943). [MF 10002]

The paper is devoted to a discussion of the weak convergence of a sequence x_n in Hilbert space to a point α and to a characterization of various types of weak convergence. It is first shown that x_n converges if and only if the difference

$|x_n - x|^2 - |x_n|^2$ converges for each x by observing that this difference is equal to $|x|^2 - 2R(x, x)$. The characterization of modes of weak convergence is made by considering the two functions $d_1(x) = \liminf |x_n - x|$, $d_2(x) = \limsup |x_n - x|$ and noting that $d_1^2(x) = d_1^2(0) - |x|^2 + |x - \alpha|^2$. It is then evident that $d_i(x)$ are continuous and have a minimum at $x = \alpha$. All weakly convergent sequences are now in one of four classes: (1) the sequence x_n is such that $0 = d_1(\alpha) = d_2(\alpha)$; (2) $0 < d_1(\alpha) = d_2(\alpha)$; (3) $0 = d_1(\alpha) < d_2(\alpha)$; (4) $0 < d_1(\alpha) < d_2(\alpha)$. If x_n is in class (1) it converges strongly. If it is in (2) it does not contain any subsequence strongly convergent to α but $|x_n - \alpha|$ converges for each x to $[d_1^2(\alpha) + |x - \alpha|^2]^{\frac{1}{2}}$. If x_n is in (3), it contains a subsequence which converges strongly to α and a subsequence x_{n_k} such that $|x_{n_k} - \alpha|$ converges to $d_2(\alpha)$; this subsequence converges weakly but not strongly to α . One has

$$d_2(x) - d_1(x) = [d_2^2(\alpha) + |x - \alpha|^2]^{\frac{1}{2}} - |x - \alpha|,$$

which attains its maximum $d_2(\alpha)$ at $x = \alpha$. If x_n is in class (4) it contains no strongly convergent subsequence and

$$d_2(x) - d_1(x) = [d_2^2(\alpha) + |x - \alpha|^2]^{\frac{1}{2}} - [d_1^2(\alpha) + |x - \alpha|^2]^{\frac{1}{2}} > 0$$

attains its maximum at $x = \alpha$. The paper closes with some examples of sequences in each class. H. H. Goldstine.

Julia, Gaston. Sur les systèmes duaux de vecteurs dans l'espace hilbertien. C. R. Acad. Sci. Paris 216, 324-326 (1943). [MF 10014]

Julia, Gaston. Sur la structure des systèmes duaux dans l'espace hilbertien. C. R. Acad. Sci. Paris 216, 396-399 (1943). [MF 10020]

Julia, Gaston. Exemples de structure des systèmes duaux de l'espace hilbertien. C. R. Acad. Sci. Paris 216, 465-468 (1943). [MF 10025]

In these three papers the author is concerned with a biorthogonal set A_i, B_j [$(A_i, B_j) = \delta_{ij}$] of points in a Hilbert space H and with the linear closed manifolds $V = [A_1, A_2, \dots]$, $W = [B_1, B_2, \dots]$ determined by these sequences. Let e_i be a complete orthonormal base for V and let A^*x, Bx be defined as $\sum e_i(x, A_i)$ and $\sum B_j(x, e_j)$, respectively. It is then shown that A^*x is defined for x in W and that $A^*Bx = x$. Moreover, the adjoints of A^* and B are $Ax = \sum A_i(x, e_i)$ and $B^*x = \sum e_j(x, B_j)$. A necessary and sufficient condition that B be a bounded operator is that the set of functional values of A^* is H . Let V be the entire space and let V_n, V_n', W_n, W_n' be the linear closed manifolds $[A_1, \dots, A_n]$, $[A_{n+1}, \dots]$, $[B_1, \dots, B_n]$, $[B_{n+1}, \dots]$. Then V_n' and W_n can be seen to be orthogonal complements; moreover, V_n and W_n' are orthogonal but are complements if and only if $W = V$. If the intersection N of all the sets V_n' contains only the origin, then the equations $(X_i, B_j) = \delta_{ij}$, regarded as determining the sequence X_i , have but one solution $X_i = A_i$. If, however, the set N contains more than the origin, then $W = H - N$ and N, W are orthogonal complements. Hence, if A_i is expressed as $n_i + w_i$, with n_i in N and w_i in W , then clearly $(W_i, B_j) = (A_i, B_j) = \delta_{ij}$ and there are no other sequences X_i such that $(X_i, B_j) = \delta_{ij}$. It is next indicated that $\sum^n (x, A_i) P_n B_i = P_n x$ and $\sum^n (x, A_i) [B_i - P_n B_i]$, where P_n is the projection of V_n , converge strongly to x . Moreover, $P_n x = \sum^n (x, B_j) P_n w_j = \sum^n (x, B_j) P_n A_j$ converges strongly to $P_W x$, where P_n, P_W are the projections of V_n and W , respectively. The second paper closes with the theorem that if x is such that the lower limit of $\|\sum^n (x, A_i) B_i\|$ is finite then x is in W ; moreover, if the sequence b_i of numbers is such that the lower limit of $\|\sum^n b_i B_i\|$ is finite, then

there is a unique x in W for which the moment problem $(A_i, x) = b_i$ has a solution. Then x_i is the weak limit of a subsequence of $\sum^n b_i B_i$.

If A is a bounded operator of the first class [C. R. Acad. Sci. Paris 213, 5-9 (1941); these Rev. 5, 100] and if $A = Ae_i$, then $\sum (x, A_i) B_i$ and $\sum (x, B_i) A_i$ converge strongly for every x , their sum is x , and $W = V = H$. If A is of the third class, the series above converge strongly to $P_V x$ and $W = V \neq H$. The third paper consists primarily of examples illustrative of the results of the other two papers.

H. H. Goldstine (Aberdeen, Md.).

Sanvicens, Francisco. On n -linear analytic functionals.

Revista Mat. Hisp.-Amer. (4) 3, 129-136 (1943). (Spanish) [MF 8951]

The author studies functionals $F[y_1(t_1), \dots, y_n(t_n)]$ which are linear in each $y_i(t_i)$, the latter being analytic in a suitable region. He shows that $F[y_1, \dots, y_n]$ may be calculated by means of an iterated Cauchy integral applied to

$$F\left[\frac{1}{\alpha_1 - t_1}, \dots, \frac{1}{\alpha_n - t_n}\right] y_1(\alpha_1) \cdots y_n(\alpha_n).$$

Infinite series expansions are given for these functionals.

E. R. Lorch (New York, N. Y.).

Nikolsky, S. Linear equations in normed linear spaces.

Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 147-166 (1943). (Russian. English summary) [MF 9882]

Let E be a complex normed vector space; let I be the identity transformation in E , B a transformation having an inverse, V a completely continuous transformation, and K a transformation such that $K(E)$ is finite dimensional. It is well known (and was first shown, independently, by Schauder and Hildebrandt) that the nondeterminantal results of the Fredholm integral equation theory can be formulated and proved for transformations in E of the form $T = I + V$. It is shown here that necessary and sufficient conditions for these theorems to hold are any of the following: (1) $T = B + V$, (2) $T = B + K$, (3) $\bar{T} = B^* + V^*$, (4) $\bar{T} = B^* + K^*$ (B^* , V^* and K^* are defined in \bar{E} but need not be adjoints of transformations in E).

The author considers next the analytic properties of the resolvent transformation $T_\lambda = I - \lambda T$. Let ϕ_T be the region in the complex plane of values λ for which $T_\lambda = B + V$ (or $B + K$), M_T the set of λ 's belonging to a component of ϕ_T containing a value in the resolvent set, and G_T the resolvent set (those λ 's for which T_λ^{-1} exists). These sets are all open and not empty. It is shown that the points of $M_T - G_T$ are isolated and that, for λ in a neighborhood of $\lambda_0 \in M_T - G_T$, $T_\lambda = \sum_{k=0}^{\infty} C_k(\lambda - \lambda_0)^k$, where the C_k for $k < 0$ have finite dimensional ranges. It is also shown that M_T consists of just those λ 's for which $T = U + V$, where $I - \lambda U$ has an inverse and $UV = 0$. Following Radon [Akad. Wiss. Wien, S.-B. IIa. 128, 1083-1121 (1919)], the Fredholm radius r_T is defined as the least upper bound of values r for which $T = U + V$, where $I + \lambda U + \lambda^2 U^2 + \dots$ converges for $\lambda < r$. It follows that r_T is the radius of the largest circle with center at $\lambda = 0$ contained in M_T . It is shown that $r_T = (r_T)^*$ from which it follows that (1) if T^* is completely continuous for some $n > 0$, then $r_T = \infty$; (2) if there exists V and $n > 0$ such that $\|T^n - V\| < 1$, then $r_T > 1$. These are previous results of the author and Yosida, respectively. The methods of proof for the theorems concerning resolvents follow closely methods introduced by F. Riesz and Radon

for special spaces; some of the theorems have already been stated by Hildebrandt [Bull. Amer. Math. Soc. 37, 196-201 (1931)].

J. V. Wehausen (Columbia, Mo.).

Hamburger, Hans Ludwig. Contributions to the theory of closed Hermitian transformations of deficiency index (m, m) . Ann. of Math. (2) 45, 59-99 (1944). [MF 9835]

This paper gives the promised proofs of some new theorems published in a note under the same title [Quart. J. Math., Oxford Ser. 13, 117-128 (1942); these Rev. 5, 40] and in addition considers the case $m = \infty$ previously omitted. The author first gives in an introduction an abstract of J. von Neumann's theory of closed Hermitian transformations of deficiency index (m, m) [written for brevity c. H. t. of d. i. (m, m)] in an Hilbert space \mathfrak{H} [J. von Neumann, Math. Ann. 102, 49-131 (1929)]. His main results follow from analytic developments based on the following theorem. "If H is a c. H. t. of d. i. (m, m) , $\phi_1(x), \phi_2(x), \dots, \phi_m(x)$ an orthonormal set of characteristic solutions of the adjoint transformation H^* which belong to the characteristic value x , then the resolvent $R_x = \hat{H}_x^{-1}$ of any self adjoint extension \hat{H} of H and the set $\phi_\mu(x)$ determine an m th order matrix $C(x) = (c_{\mu\nu}(x))$ and its inverse $C^{-1}(x) = (c^{-1}(x))$ for every nonreal x such that

$$(x-y)R_x\phi_\mu(x) = \phi_\mu(x) - \sum_{\nu=1}^m \sum_{\sigma=1}^m c_{\mu\nu}^{-1}(x)c_{\sigma\nu}^{-1}(x)\phi_\sigma(y)$$

for every nonreal x and y . For $m = \infty$ the two matrices $C(x)$ and $C^{-1}(x)$ are uniformly bounded in every closed domain of x which does not contain any point of the real axis." The author then shows that the m functions $\Phi_\mu(x)$ defined by

$$\Phi_\mu(x) = \sum_{\nu=1}^m c_{\mu\nu}^{-1}(x)\phi_\nu(x), \quad \mu = 1, 2, \dots, m,$$

depend analytically on x , and obtains analytic representations for these functions both when \hat{H} has a simple and when it has a multiple spectrum.

After defining a prime transformation as a c. H. t. H of d. i. (m, m) if \mathfrak{H} does not contain any closed linear manifold which reduces H and with respect to which H is self-adjoint, he gives a test for prime transformations and completely solves the problem of constructing all c. H. prime transformations H of which a given self-adjoint transformation \hat{H} is an extension. In particular, if \hat{H} has a simple spectrum, he shows that there exist prime transformations H of d. i. (m, m) for all values of $m = 1, 2, \dots$ and $m = \infty$ but, if \hat{H} has a spectrum of multiplicity k , only for values of $m \geq k$.

In the final part of the paper, concerned with the nature of the spectrum of a self-adjoint extension of a c. H. prime transformation, the author shows that the spectrum of every self-adjoint extension of a c. H. prime transformation H of d. i. (m, m) , where m is finite, consists of an infinite number of isolated characteristic values if, and only if, there exists an orthonormal set of m characteristic solutions $\phi_\mu(x)$ of H_x^* continuous in the whole x -plane, that is, continuous even for real values of x . If a single self-adjoint extension of a c. H. prime transformation H of d. i. (m, m) , where m is finite, has a point spectrum only, without continuous spectrum, then so has every self-adjoint extension of H .

(The corresponding result when $m = \infty$ will be given in a later paper.)

Throughout the author uses for purposes illustrative of his theory the ordinary linear self-adjoint differential equation of the second order

$$L(f) = -(d/dt)^2[p(t)f(t)] + q(t)f(t) = 0,$$

where p and q are real continuous functions in the interval $a \leq t \leq b$ and where $p > 0$ in the same interval. As a final illustration he shows that if x_1 and x_2 are two real numbers there are always real self-adjoint boundary conditions such that x_1 and x_2 are characteristic values of $L(f)$ which correspond to these boundary conditions. J. Williamson.

Friedrichs, K. O. The identity of weak and strong extensions of differential operators. Trans. Amer. Math. Soc. 55, 132-151 (1944). [MF 9878]

Let R be an open region in Euclidean m -space and let $u(x)$ denote a real vector function with s components defined on R . Let \mathfrak{E} and \mathfrak{D} denote the classes of vector functions which are continuous and continuously differentiable, respectively, in R . Let A_1, \dots, A_m and B be matrices with t rows and s columns. They are assumed to be real continuous functions of x in R , and the A 's are assumed to be continuously differentiable. The differential operator E to be considered carries functions u of class \mathfrak{D} into functions

$$v = A_s \frac{\partial u}{\partial x_s} + Bu$$

(μ summed from 1 to m). Of course, v is a vector function with t continuous components. The formal adjoint of E is denoted by E^* . It carries functions with t components into functions with s components. A function which vanishes outside of some bounded closed subset of R is denoted by \dot{u} . The symbols \mathfrak{E} , \mathfrak{D} denote the subclasses of \mathfrak{E} and \mathfrak{D} , respectively, consisting of such functions. Let $\dot{u}u = \dot{u}_\sigma u_\sigma$ (sum on σ from 1 to s), and

$$(\dot{u}, u)_R = \int_R \dot{u}u dx$$

when \dot{u} is in \mathfrak{E} and u is in \mathfrak{E} . The space \mathfrak{L}_p ($p \geq 1$) is the space of all measurable functions for which the norm

$$\|u\|_p = \left[\int_R |u|^p dx \right]^{1/p}$$

exists, where

$$|u| = \left[\sum_\sigma |u_\sigma|^2 \right]^{1/2}.$$

If $p = \infty$, suitable modifications are made in these definitions. The class \mathfrak{E} is dense in \mathfrak{L}_p . The symbols \mathfrak{E} , \mathfrak{D} , \mathfrak{L}_p , etc. are used to refer to vector functions with either s or t components, according to the context.

The weak extension of E is defined as follows. By \mathfrak{G} denote the class of functions u in \mathfrak{L}_p corresponding to which there exists a function v in \mathfrak{L}_p such that for all \dot{u} in \mathfrak{D} we have $(E^*\dot{u}, u)_R = (\dot{u}, v)_R$. Then define $v = Eu$. It is proved that \mathfrak{G} includes \mathfrak{D} , and that we have an extension of E in the usual sense of the word extension. The strong extension of E is defined as follows. A proper subregion R' of R

is an open region contained in a bounded closed subset of R . A function u in \mathcal{E} , is said to belong to class \mathfrak{F} if there exists a function v in \mathcal{E} , such that, corresponding to each proper subregion R' there is a family u_ϵ of functions in \mathfrak{D} for which $\|u_\epsilon - u\|_{R'} \rightarrow 0$ and $\|E u_\epsilon - v\|_{R'} \rightarrow 0$ as $\epsilon \rightarrow 0$. Then define $v = Eu$.

The main theorem of the paper is that the weak and strong extensions of E are coincident (in particular, $\mathfrak{F} = \mathfrak{G}$). This coincidence is also proved for a more general type of norm. In case the region R and the matrices A_μ, B satisfy certain restrictive conditions, the limitation to proper subregions R' in the definition of the strong extension may be removed. The main theorem is proved by constructing a family of integral operators J_ϵ such that $u_\epsilon = J_\epsilon u$ furnishes the family of functions called for in the definition of the strong extension. A special result of the paper is that, if R is a simply connected two dimensional region and $u = \{u_1, u_2\}$ is a vector for which $\text{div } u = 0$ (weak extension of the divergence), then $u = \text{curl } v$ for some v (strong extension of the curl). The paper also contains theorems about adjoint operators and about undetermined systems. *A. E. Taylor.*

Hille, Einar. On the theory of characters of groups and semi-groups in normed vector rings. *Proc. Nat. Acad. Sci. U. S. A.* **30**, 58-60 (1944). [MF 10127]

If an Abelian group \mathfrak{G} is imbedded in a normed ring, Gelfand has shown [Rec. Math. [Mat. Sbornik] N.S. **9**(51), 49-50 (1941); these Rev. **3**, 36] that the functions $\mu(x, \mathfrak{M}) = \alpha\epsilon$ (defined by $x = \alpha\epsilon(\mathfrak{M})$ where $x \in \mathfrak{G}$ and \mathfrak{M} is a maximal ideal) are characters which are distinct (that is, $x \neq y$ implies $\mu(x, \mathfrak{M}) \neq \mu(y, \mathfrak{M})$) if the cyclic subgroups of \mathfrak{G} are of bounded norm. The author obtains this result for the weakened hypothesis $\|x^{n^2}\| = o(n)$. An easy example shows that this cannot be improved. The proof uses a result of Pólya on functions $f(z)$ which are entire in $w = (z-1)^{-1}$.

E. R. Lorch (New York, N. Y.).

Halmos, Paul R. Approximation theories for measure preserving transformations. *Trans. Amer. Math. Soc.* **55**, 1-18 (1944). [MF 9872]

Let B be a Boolean σ -algebra, such as, for example, the measurable sets modulo sets of measure zero in a measure space. If the usual notations are employed and addition and multiplication are defined by the formulas $a+b = (a \cap b') \cup (a' \cap b)$, $ab = a \cap b$, then B is an algebraic ring. Let there be defined on B a numerically valued, positive, countably additive, finite measure denoted by $|a|$, where $a \in B$. It is known that the existence of this measure implies that B is complete in the sense of lattice theory. Let the distance between a and b be defined to be $|a-b|$. It can be shown that B is then a complete metric space. It is furthermore assumed that $|1| = 1$, B is nonatomic and B is separable. An automorphism of B is a one-to-one mapping T of B onto itself such that $|Ta| = |a|$, $Ta' = (Ta)'$, $T(\cup a_i) = \cup Ta_i$, $T(\cap a_i) = \cap Ta_i$. The set of all automorphisms form a group G and let $N(S; a, \epsilon)$, where $S \in G$, $a \in B$, $\epsilon > 0$, denote the set of all elements T of G such that $|Sa - Ta| < \epsilon$. If the sets $N(S)$ are used as a subbase of the open sets, a topology is defined in G which is called the neighborhood topology of G . The algebra B admits a representation on the measurable sets, mod sets of measure zero, of a unit interval and, if this interval is suitably divided

into subintervals and these subintervals are permuted, there is thereby defined a permutation on B . The automorphism T is ergodic if $Ta = a$ implies $a = 0$ or $a = 1$. The following interesting results are shown to be true in the neighborhood topology: (1) G is a complete topological group satisfying the second countability axiom; (2) the permutations and even the cyclic permutations are dense in G ; (3) the ergodic automorphisms are dense in G ; (4) the set of ergodic automorphisms is a residual G_δ . Thus further light is shed on the question of the generality of ergodic measure preserving transformations.

Two other stronger topologies are introduced into G . If E is any subset of B , let E^* denote the set of all those elements of B , every nonzero subelement of which contains a nonzero subelement belonging to E . (Among other results it is shown that E^* is a principal ideal.) Given S and T in G , let $E(S, T)$ be the set of all a in B such that $Sa \neq Ta$ and let the distance $d(S, T) = |\sup E^*(S, T)|$ define the metric topology in G . In this metric topology, G is a complete topological group and $d(S, T)$ is invariant under both right and left translations. In this topology the periodic automorphisms are dense in G , but the set of ergodic automorphisms is nowhere dense in G . A third topology, the uniform topology, is introduced into G by defining the distance between S and T of G to be $\delta(S, T) = \sup \{|Sa - Ta| : a \in B\}$. But it turns out that $(2/3)d(S, T) \leq \delta(S, T) \leq d(S, T)$, and consequently the metric and uniform topologies are equivalent. *G. A. Hedlund (Charlottesville, Va.).*

Fréchet, Maurice. Sur le théorème ergodique de Birkhoff. *C. R. Acad. Sci. Paris* **213**, 607-609 (1941). [MF 9642]

In a preceding note [C. R. Acad. Sci. Paris **213**, 520-522 (1941); these Rev. **5**, 96] the author defined an asymptotically almost periodic continuous (A.P.P.C.) function as the sum of a Bohr almost periodic function and a continuous function which approaches zero at infinity (and also gave other equivalent definitions). In the present paper the author deals with a family of transformations T_t which take each point of a bounded closed set S in m -dimensional Euclidean space into a point of S . As his weaker hypothesis, he assumes that $T_{t+t'}M = T_t(T_{t'}M)$ for all M on S , $t \geq 0$, $t' \geq 0$, and that T_tM is a continuous function of t for each M and an equally continuous function of M for all $t \geq 0$. Under this hypothesis, he states that, for each fixed M , T_tM is an m -dimensional A.P.P.C. function of t and that, if $f(M)$ is continuous on S , $f(T_tM)$ is also an A.P.P.C. function of t . As a consequence, the mean

$$\lim_{L \rightarrow \infty} (1/L) \int_0^L f(T_tM) dt$$

exists for each M of S . Finally, if a law of probability is introduced and if S is indecomposable with respect to that law, the author states that the above average is independent of M and is equal to a certain spatial average

$$\int_S f(M) du(e),$$

where $u(e)$ is an additive set-function of subsets of S which is independent of f and equal to unity when $e = S$.

R. H. Cameron (Cambridge, Mass.).

MECHANICS

Parodi, Hippolyte. Calcul des lignes caténares inclinées. C. R. Acad. Sci. Paris 216, 28-29 (1943). [MF 9999]

The author linearizes a catenary problem by an additional assumption and describes the solution in terms of tabulated particular solutions. *P. Franklin.*

Gutiérrez Novoa, Lino. On the formulas of relative movement. Revista Soc. Cubana Ci. Fis. Mat. 1, 105-108 (1943). (Spanish) [MF 10109]

Brelot, M. Sur quelques points de mécanique rationnelle. Ann. Univ. Grenoble 20, 1-37 (1944). [MF 10119]

Continuation of the author's development of mechanics [Ann. Univ. Grenoble 19, 24 pp. (1943); these Rev. 5, 16]. This deals largely with questions of existence and stability for simple cases where the results follow directly from known theorems of analysis. *P. Franklin* (Cambridge, Mass.).

Platrier, Charles. Extension des équations de Lagrange et d'Appell à des systèmes soumis à des liaisons non holonomes plus complexes que les liaisons de Neumann. C. R. Acad. Sci. Paris 216, 369-371 (1943). [MF 10019]

Basing his considerations upon the least-curvature principle of Gauss and Hertz, the author gives generalizations of the Lagrangian equations and of Appell's equations which are applicable to the case of a dynamical system subject to constraints $L_i = 0$, where the L 's are arbitrary functions of the generalized coordinates, velocities and accelerations, having first partial derivatives with respect to the accelerations. *L. A. MacColl* (New York, N. Y.).

Četajev, N. G. Forced motions. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 25-30 (1943). (Russian. English summary) [MF 9722]

The author derives analogues of d'Alembert's principle, of the energy theorem and of Lagrange's equations of motion of the second kind for a mechanical system consisting of n particles subjected to workless constraints and such that to each particle is associated a parameter which changes with time according to a given differential equation of first order. *L. Bers* (Providence, R. I.).

Četajev, N. G. Concerning the sufficient conditions of the stability of a rotating motion of a projectile. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 81-96 (1943). (Russian. English summary) [MF 9729]

This paper treats the stability of a rotating projectile for the cases (1) rectilinear flight with constant velocities of flight and of rotation, (2) variable velocity of rotation, (3) variable velocities of flight and of rotation, (4) curvilinear flight and variable velocities, (5) motion on a circular path with constant velocity, (6) a more general path. In case (1) the problem is dynamically similar to that of a top on a smooth table and leads to the well-known cubic polynomial in $u = \cos \theta$ (θ the angle of nutation). The other cases are studied by means of the changes produced in this cubic by the additional forces introduced. The conditions for stability are expressed in the form of inequalities involving the constants of the trajectory, the projectile and the initial conditions. The term "stability" is not defined, but apparently means that the angle of nutation oscillates between restricted limits. *W. E. Milne* (Corvallis, Ore.).

Meyer zur Capellen, Walther. Getriebependel. II. Z. Instrumentenkunde 61, 1-14 (1941). [MF 9968]

Meyer zur Capellen, Walther. Getriebependel. III. Z. Instrumentenkunde 62, 123-138 (1942). [MF 9970]

In previous papers [Z. Instrumentenkunde 55, 393-407, 437-448 (1935)] the author has studied the small free oscillations of a four-link mechanism about the equilibrium position which the mechanism assumes under the influence of gravity. The first of the present papers deals with the case that, in addition to gravity, the restoring forces of springs attached to the two moving joints are acting on the four-link mechanism. The second paper deals with the small forced vibrations of such a mechanism. The treatment, mostly of an elementary character, is exhaustive. *W. Prager* (Providence, R. I.).

Horvay, G. and Pines, S. Pi-Tee transformations in the analysis of mechanical transmission lines. J. Appl. Mech. 11, A-41-A-46 (1944). [MF 10090]

The paper gives a direct proof of the equivalence of a system consisting of a mass between two springs to a suitable system of two masses joined by one spring and illustrates the importance of this principle by several examples. *I. Opatowski* (Chicago, Ill.).

Astronomy

Sémirot, Pierre. Sur les mouvements périodiques d'une corps attiré par deux centres fixes. C. R. Acad. Sci. Paris 215, 408-411 (1942). [MF 9509]

The problem of motion of a particle in space under the attraction of two fixed gravitational centers is one whose solutions have long been known in the form of quadratures involving elliptic functions [see P. Appell, *Mécanique Rationnelle*, vol. I, Gauthier-Villars, Paris, 1926, pp. 585-591]. In the present paper the conditions that an orbit be periodic are given in terms of two linear relations between the periods of the elliptic integrals. [Similar results for the planar problem are given by C. L. Charlier, *Mechanik des Himmels*, vol. I, Veit & Co., Leipzig, 1902, pp. 117-163; see also A. M. Hildebrandt, *Amer. J. Math.* 33, 337-362 (1911).] The stability of periodic orbits is investigated by means of characteristic exponents and application of a criterion of Poincaré [see E. T. Whittaker, *Analytical Dynamics*, 4th ed., Cambridge University Press, 1937, p. 400] shows that all such orbits are unstable. *W. Kaplan* (Ann Arbor, Mich.).

Sokoloff, G. Sur les trajectoires de collision simultanée de trois points matériels dont les forces attractives ou répulsives dépendent de leurs distances mutuelles. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 112-115 (1941). [MF 9625]

This paper is a continuation of previous ones by the author on the collisions of three bodies which attract or repel each other under a general force depending on the distance [see Acad. Roy. Belgique. Bull. Cl. Sci. (5) 22, 295-305, 540-551 (1936)]. Results are stated for the case when the sum of the mutual distances approaches zero for a finite value t_1 of the time. As t approaches t_1 the three bodies approach collision, the limiting configuration being

an equilateral triangle or a collinear configuration. Indications are given of the form of series expansions in the neighborhood of collision. *W. Kaplan* (Ann Arbor, Mich.).

Buchanan, Daniel. Periodic orbits for four finite bodies with repulsive and attractive forces. *Trans. Roy. Soc. Canada. Sect. III.* 37, 1-7 (1943). [MF 9972]

The author considers the motions of a system of four equal particles P_{ij} ($i, j=1, 2$) such that the P_{1j} attract the P_{2k} ($j=1, 2; k=1, 2$) while P_{1i} and P_{2i} ($i=1, 2$) repel each other, the forces being proportional to the inverse square of the distance (with different proportionality constants for attraction and repulsion). The P_{1i} are constrained to move on the z -axis, at equal distances above and below the xy -plane; the P_{2i} are constrained to move in the xy -plane symmetrically with respect to the origin. Particular periodic solutions are found in which the P_{1i} are fixed and the P_{2i} move in circles about the origin. These are extended to a larger family of nearby periodic solutions by the method of expansion in power series in a parameter.

W. Kaplan (Ann Arbor, Mich.).

Buchanan, Daniel. Periodic and asymptotic orbits in a five body problem. *Canadian J. Research. Sect. A.* 22, 1-25 (1944). [MF 9866]

The author considers the restricted five-body problem in which four equal finite masses revolve, in the form of a square, about their center of gravity, while an infinitesimal mass moves in space subject to their attraction. The libration points for this case were found by Hinrichsen [*Amer. Math. Monthly* 50, 231-237 (1943); these Rev. 4, 227]. The author determines formal periodic and asymptotic solutions near the libration points as expansions in powers of a parameter ϵ [cf. F. R. Moulton, *Periodic Orbits*, Carnegie Institute, Washington, 1920, chap. 5], the convergence of the series being a consequence of general theorems of Poincaré and MacMillan. The periodic solutions include three-dimensional ones, for which one approximate orbit is computed and graphed.

W. Kaplan.

Cesco, Reynaldo P. The secular perturbations of Pluton. *Observ. Astron. Univ. Nac. La Plata. Serie Astron.* 17, 5-69 (1941). (Spanish) [MF 10133]

Wilkens, Alexander. Die Säkularbeschleunigung der grossen Achsen der Planetenbahnen. *Observ. Astron. Univ. Nac. La Plata. Serie Astron.* 18, 5-186 (1942). (German and Spanish) [MF 10132]

García, Godofredo. Motion of a continuous system under the influence of gravitation only. Application to nebulae. *Revista Ci., Lima* 45, 463-483 (1943) = *Actas Acad. Ci. Lima* 6, 135-155 (1943). (Spanish) [MF 10100]

The present paper is based on a paper by Levi-Civita [*Scritti Mat. Off. a L. Berzolari, Pavia, 1936*, pp. 161-168] on the same topic. The author has found an error in sign in one of Levi-Civita's equations (equation (7), p. 165) and has carried through the solution with the correction made. The analytical form of the result is now quite different.

W. Kaplan (Ann Arbor, Mich.).

Chandrasekhar, S. New methods in stellar dynamics. *Ann. New York Acad. Sci.* 45, 133-161 (1943).

This paper is a synthesis of recent work by the author [see, for example, *Astrophys. J.* 94, 511-524 (1941); 97,

255-273 (1943); 98, 54-60 (1943); these Rev. 3, 281; 4, 260; 5, 18] and by von Neumann and the author [*Astrophys. J.* 95, 489-531 (1942); these Rev. 3, 281] on the statistics of the fluctuating gravitational field acting on a star. Elaborate mathematical details are avoided and the emphasis is on the fundamental reasoning. To elucidate the theory a paragraph-by-paragraph comparison of analytical dynamics and statistical dynamics is given. *W. Kaplan.*

Chandrasekhar, S. The statistics of the gravitational field arising from a random distribution of stars. III. The correlations in the forces acting at two points separated by a finite distance. *Astrophys. J.* 99, 25-46 (1944). [MF 9869]

Continuing his earlier investigations into the statistical variation of the gravitational force on a star due to a Poisson spatial distribution of stars [Chandrasekhar and von Neumann, *Astrophys. J.* 95, 489-531 (1942); 97, 1-27 (1943); these Rev. 3, 281; 4, 227], the author now discusses the simultaneous distribution of the gravitational forces F_P, F_Q at two points P, Q . Explicit evaluations of the average value of F_P for given F_Q and related quantities are obtained.

J. L. Doob (Washington, D. C.).

Chandrasekhar, S. The statistics of the gravitational field arising from a random distribution of stars. IV. The stochastic variation of the force acting on a star. *Astrophys. J.* 99, 47-53 (1944). [MF 9870]

Continuing his earlier investigations [cf. the preceding review], the author now discusses the simultaneous distribution of F_0 and F_t (where F_t is the force on a star at time t). At this level of approximation, it is supposed that the stars are moving linearly and independently with Gaussian velocity distributions. Evaluations of the average value of F_t for given F_0 and related quantities are obtained.

J. L. Doob (Washington, D. C.).

Gleissberg, W. Integral principles of stellar equilibrium. *Rev. Fac. Sci. Univ. Istanbul. Ser. A.* 7, 12-19 (1942). (English. Turkish summary) [MF 9761]

For a spherically symmetric stellar configuration in equilibrium

$$\frac{dP}{dr} = -G \frac{M(r)}{r^2} \rho,$$

where P denotes the total pressure, ρ the density, G the constant of gravity and $M(r)$ the mass interior to the point distant r from the center. Such an equilibrium configuration is compared with a nonequilibrium configuration in which the pressure distribution is allowed to be arbitrary except for the restriction that P is to have the same values at the center and the boundary as the equilibrium configuration. However, the density distribution in the nonequilibrium configuration is assumed to be identical with that in the equilibrium configuration. It is then shown that the integral

$$\int \left(2M(r) + \frac{r^2}{G\rho} \frac{dP}{dr} \right) dP$$

extended from the boundary to the center attains its maximum value for the equilibrium configuration. A similar theorem is proved for stars in radiative equilibrium.

S. Chandrasekhar (Williams Bay, Wis.).

Sen, N. R. Contribution to the theory of stellar models. Proc. Nat. Inst. Sci. India 8, 339-360 (1942). [MF 10032]

Stellar models in which the ratio η of the average rate of generation of energy interior to the point r , to the average for the whole star, varies as $\rho^\alpha T^\beta$ (where ρ denotes the density, T the temperature, and $\alpha, \beta > 0$) are considered from the point of view of the stability against convection of the assumed radiative temperature gradient. In the discussion a law of opacity of the form $\kappa \propto \rho^b T^{-s-\delta}$ is assumed. It is known that physically the most significant of the solutions of the equations of equilibrium are those for which ρ and T tend to zero simultaneously and that for these solutions

$$(1) \quad \rho \propto T^{(6+\alpha-\delta)/(\alpha+\beta+1)}, \quad \rho, T \rightarrow 0$$

[S. Chandrasekhar, Introduction to the Study of Stellar Structure, University of Chicago Press, Chicago, 1939, pp. 327-330, eq. (64)]. On the other hand for the stability of the radiative gradient it is necessary that the effective polytropic index be greater than $\frac{3}{2}$. This implies according to (1) that $(6+\alpha-\delta) > 3(\alpha+\beta+1)/2$. This is the principal result contained in this paper. S. Chandrasekhar.

Sen, N. R. On the inversion of density gradient and convection in stellar bodies. Proc. Nat. Inst. Sci. India 7, 183-196 (1941). [MF 10033]

In the first part of this paper certain results of Chandrasekhar [Z. Astrophys. 14, 164-188 (1937)] and Severny [Monthly Not. Roy. Astr. Soc. 97, 699-704 (1936)] on the inversion of density gradients in stellar models are derived by somewhat different methods. In the second part these methods are extended to discuss the stability of the assumed radiative temperature gradients [see the preceding review]. S. Chandrasekhar (Williams Bay, Wis.).

Sen, N. R. On stellar models based on Bethe's law of energy generation. Proc. Nat. Inst. Sci. India 8, 317-330 (1942). [MF 10031]

In this paper the equations of hydrostatic and radiative equilibrium of the envelope of a point source stellar model with a law of opacity $\kappa \propto \rho^a T^{-b}$ ($a, b > 0$) are discussed and it is shown that finite stellar configurations are possible only when $b < 4a$. [It should perhaps be pointed out that this result is contained in certain earlier investigations by J. Tuominen, Ann. Acad. Sci. Fennicae (A) 48, no. 16 (1938); Ann. New York Acad. Sci. 41, 61-76 (1941).] S. Chandrasekhar (Williams Bay, Wis.).

Tuominen, Jaakko. Über den inneren Aufbau der Trümplerschens Sterne. Z. Astrophys. 22, 90-110 (1943). [MF 9904]

Ambarzumian, V. On the scattering of light by the planetary atmospheres. Astr. J. Soviet Union [Astr. Zhurnal] 19, no. 5, 30-41 (1942). (Russian. English summary) [MF 9964]

The author obtained a sufficiently exact solution of the problem in the form of a functional equation easily calculated numerically. The character of the indicatrix of scattering may be arbitrary. In the present work only the spherical indicatrix is considered. The application is made to the problem of the distribution of brightness over the planetary and the solar discs. Author's summary.

Hydrodynamics, Aerodynamics, Acoustics

Poncin, Henri. Sur une méthode de prolongement analytique applicable à divers problèmes d'hydro- et d'aérodynamique. C. R. Acad. Sci. Paris 213, 341-342 (1941). [MF 9163]

The author states that there exist many questions in fluid dynamics which can be formulated as problems in functional analysis and indicates a method for treating such problems. Some of the symbols which are introduced are not explained. The problem and the procedure are described in a general and vague manner, making it impossible for the reviewer to obtain a clear understanding of the results. S. Bergman (Providence, R. I.).

Jacob, Caius. Sur l'écoulement fluide produit par la rotation d'un biplan "en tandem" autour d'un axe-situé dans son propre plan. Mathematica, Timișoara 19, 106-118 (1943). [MF 9958]

The paper is concerned with the two-dimensional flow of incompressible nonviscous fluid around two flat plate airfoils of infinite span and placed in tandem, rotating about an axis parallel to the span. The fluid very far from the airfoils is assumed to be at rest. The problem is reduced to a Dirichlet problem with mixed boundary conditions which specify the value of the potential function on the line joining the chords of the airfoils and the value of the stream function on the chords of the airfoils. This problem is then solved by the method of A. Signorini [Ann. Mat. Pura Appl. (3) 25, 253-273 (1916)]. The resultant force and the moment about the axis of rotation are also determined. For the particular case of two airfoils of equal chord c placed symmetrically with respect to the center of rotation, the resultant force is found to be zero but the moment is equal to $-(\pi/2)\rho\omega c^2 d^2$. Here ρ is the density of the fluid, ω the angular acceleration and d the distance from the center of rotation to the centers of the airfoils. H. S. Tsien (Pasadena, Calif.).

Kravtchenko, Julien. Sur un principe de minimum dans l'hydrodynamique des fluides visqueux. C. R. Acad. Sci. Paris 213, 977-980 (1941). [MF 9665]

In the introduction the author states that he has shown the impossibility of forming an integral functional, quadratic in terms of the velocity components u, v, w , such that the equations of Euler for the stationary value of the functional are the partial differential equations for the flow of viscous fluids. He then treats the more restricted problem of slow steady movement (inertia forces neglected) of viscous fluids. If μ is the coefficient of viscosity, ρ the density, p the pressure, and U the potential of external forces, also independent of time, the partial differential equations are

$$\begin{aligned} \mu \Delta u + (\partial/\partial x)(\rho U - p) &= 0, \\ (1) \quad \mu \Delta v + (\partial/\partial y)(\rho U - p) &= 0, \\ \mu \Delta w + (\partial/\partial z)(\rho U - p) &= 0 \end{aligned}$$

and

$$(2) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Consider a domain D bounded by a closed surface S which is sufficiently regular to allow the application of Green's theorem. The function U is supposed to have continuous first derivatives in the closed domain $D+S$. The problem is to solve (1) and (2) such that u, v, w and p assume pre-assigned values on S . The author shows that the connected

variational problem is to find the stationary value of the following integrals:

$$(3) \quad I(p) = \int_D \int \left[\left(\frac{\partial p}{\partial x} \right)^2 + \left(\frac{\partial p}{\partial y} \right)^2 + \left(\frac{\partial p}{\partial z} \right)^2 \right] dx dy dz,$$

$$(4) \quad J(u, v, w) = \int_D \int \left[\Psi(u, v, w) - u \frac{\partial}{\partial x} (\rho U - p) - v \frac{\partial}{\partial y} (\rho U - p) - w \frac{\partial}{\partial z} (\rho U - p) \right] dx dy dz,$$

where $\Psi(u, v, w)$ is the dissipation function of Lord Rayleigh. *H. S. Tsien* (Pasadena, Calif.).

Price, H. L. The lateral stability of aeroplanes. VI. Derivative calculation by the Lotz system. *Aircraft Engrg.* 15, 345-351 (1943). [MF 9881]

[The previous articles appeared in *Aircraft Engrg.* 15, 193-198, 228-233, 265-269, 281-287, 325-329; these *Rev.* 5, 81, 136.]

The Prandtl lifting-line wing theory is employed to evaluate the lift, induced-drag, rolling-moment and yawing-moment coefficients for wings having angular velocities in rolling and yawing, respectively. Formulas of general validity are thereby deduced, involving the Fourier coefficients of the circulation distribution. It is implicitly assumed that the Prandtl theory is applicable to such essentially non-stationary situations. *W. R. Sears* (Inglewood, Calif.).

Kozlov, V. S. On the design of water flowing under structures in strata of different permeability. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 32, 536-539 (1941). [MF 9609]

This note deals with the plane movement of ground water under hydraulic structures if the underground is composed of two strata of different permeability. It is assumed that none of the rows of sheet piles reaches down to the division line between the two strata which is assumed to be horizontal. The lower stratum can be either limited in thickness or infinitely thick. The solution is an analytic one. The main idea is that the velocity distribution in both strata can be expressed in terms of the distribution of the complex flow potential along the division line. The author deduces two singular integral equations, the solution of which gives this distribution. *P. Neményi* (Pullman, Wash.).

Görtler, H. Über eine Schwingungserscheinung in Flüssigkeiten mit stabiler Dichteschichtung. *Z. Angew. Math. Mech.* 23, 65-71 (1943). [MF 9865]

The author investigates the problem of small oscillations in incompressible fluid at rest under gravitational force with a given density $\rho_0(z)$ as a function of the vertical coordinate z only. The density stratification is assumed to be stable, that is, $\rho_0'(z) \leq 0$. The oscillation is excited by a submerged body and has a circular frequency β . It is shown that the type of the partial differential equations of the small oscillation is elliptic, parabolic or hyperbolic if $\beta^2 + g\rho_0'/\rho_0$ is positive, zero or negative, respectively; here g is the gravitational constant. In the hyperbolic case, the real characteristics of the differential equations can be considered as lines of discontinuity in density. For the special case of $\rho_0 = \rho_1 e^{-z/h}$ ($\rho_1, h = \text{constant}$), $-g\rho_0'/\rho_0 = g/h$, the characteristic or lines of density discontinuity are straight lines with the inclinations φ given by $\sin \varphi = \pm \beta/(g/h)^{1/2}$. This relation is shown to agree with the experimental data obtained by exciting a salt water channel at one point near the upper

surface and observing the density variation with the Schlieren method. The critical frequency β_c corresponding to $\varphi = \varphi_c = \pi/2$ is $\beta_c = (g/h)^{1/2}$. This is the frequency of the transversal waves of Lord Rayleigh and V. Bjerknes. However, the phenomenon of density discontinuity seems to be a new one not investigated before. *H. S. Tsien*.

Maranz, M. S. La propagation des ondes de choc dans un canal. *Appl. Math. Mech.* [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 179-192 (1943). (Russian. French summary) [MF 10075]

Dans cet article on étudie les ondes longues dans un canal. L'auteur résout le problème de propagation des ondes de choc, le mouvement initial étant arbitraire. On détermine d'abord les instants, où les ondes de choc prennent naissance. Puis on forme un système des équations différentielles de ce problème. On examine qualitativement la solution de ce système et on démontre le théorème d'existence et d'unicité de résolution. *Author's summary.*

Dorodnizin, A. On the boundary layer of a compressible gas. *Appl. Math. Mech.* [Akad. Nauk SSSR. Prikl. Mat. Mech.] 6, 449-486 (1942). (Russian. English summary) [MF 8362]

The author considers the boundary layer in the case of a compressible fluid under assumption that the Prandtl number equals one and that no heat exchange between the fluid and the walls occurs. Using suitable variables the boundary layer equation is reduced to a form similar to that of an incompressible fluid:

$$d \left(\int_0^s \bar{u}^2 dt \right) / ds - U d \left(\int_0^s \bar{u} dt \right) / ds = U f(s) \int_0^s (1 - \bar{u}^2) dt - (d\bar{u}/dt)_{t=0} = 0.$$

Here s and t are the coordinates; $t=0$ and $t=\delta$ correspond to the boundary of the obstacle and the exterior boundary of the layer, respectively; u and v are the velocity components, U is the velocity of the main flow and V the speed along the exterior boundary of the layer, i_0 is the total energy, $\bar{u} = u/(2i_0)^{1/2}$ and $f(s) = [1 - V^2/2i_0]^{-1/2} dU/ds$. Following the method of von Kármán and Polhausen, the author writes the solution \bar{u} in the form $\bar{u} = \sum_{k=1}^{\infty} A_k \tau^k$, $\tau = t/\delta$, where the A_k are constants determined from the boundary conditions. He then obtains (in the usual way) the ordinary differential equation for $\delta(s)$. The case of a flat plate is discussed in detail. The considerations are generalized to the case of bodies of revolution. By the method of Blasius [*Z. Math. Phys.* 56, 1 (1908)] and of Howarth [*Aeronaut. Res. Comm., Research Mem. no. 1632* (1935)] the author determines the solution of the problem in the form of an infinite series $S = \sum_{m,n} \alpha_{mn} s^m t^n$. [Reviewer's remark. The series does not necessarily converge in the whole regularity domain (in the real s, t -plane) of the solution. If the solution is an analytic function of two real variables s and t , one can obtain, using certain summation methods, from S a representation which is valid in the whole regularity domain. For instance,

$$\lim_{h \rightarrow 0} \{ \sum_{m,n} \alpha_{mn} s^m t^n / \Gamma[1 + h(n+m)] \}$$

will represent the solution in the largest Mittag-Leffler star in the s, t -plane in which the solution is regular.]

S. Bergman (Providence, R. I.).

Grimminger, G. Velocity and mass distributions resulting from the lateral diffusion of a current in a stratified medium on a rotating earth. J. Franklin Inst. 236, 413-443, 509-520 (1943). [MF 9442]

Lateral diffusion of a rectilinear current in a stratified medium composed of two layers is discussed with the aid of the equations $0 = f v_n D_n + \partial(T_n D_n)/\partial y$, $0 = f \rho_n u_n + \partial p_n/\partial y$, $0 = \partial D_n/\partial t + \partial(v_n D_n)/\partial y$; $T_n = v_n \partial u_n/\partial y$, $v_n = v_0 H_0/D_n$, $T_n = \tau_n/\rho_n$, $n=1, 2$, where f is the parameter of Coriolis, τ_n is the shearing stress, ρ_n the density, u_n the axial component of velocity, and v_n the transverse component. As usual p denotes the pressure, t the time and v_n the kinematic coefficient of eddy viscosity, H_0 is the mean depth of the fluid and v_0 is a corresponding value of eddy viscosity. The pressure gradients in the two layers are expressed in terms of the thicknesses D_1, D_2 of the upper and lower layers, respectively, and a pair of partial differential equations for D_1 and D_2 are reduced to the form of Rossby's equation $\partial K_n/\partial s_n + \partial^2 K_n/\partial \eta^2 = 0$, $n=1, 2$; $K_1 = D_1 + k D_2$, $K_2 = D_1 - k D_2$, $s_1 = s(1+k^{-1})$, $s_2 = s(1-k^{-1})$, $k = (\rho_2/\rho_1)$, $\eta = y/\lambda$, $f\lambda = (gH_0)^{1/2}$, $s = (v_0\lambda/\lambda^2)$. A solution of Rossby's equation is obtained by means of Fourier's integral and a solution is also developed for the single layer when the initial current is of finite width. Three place tables are given for the functions $F(x) = \int_0^\infty \cos(mx) e^{-im^2} dm$, $\Phi(x) = \int_0^\infty F(t) dt$, $\Psi(x) = \int_0^\infty \Phi(t) dt$ for $x=0(0.025)2.5(0.05)8$. It is found that $F(x)=0$ when $x=2.435, 4.778$ and 6.550 . The functions $\Phi(x), \Psi(x)$ are zero for $x=0$ and positive for positive values of x ; $\Phi(x)$ has a maximum value 1.736 when $x=2.435$ and a minimum value 1.549 when x is about 4.7.

In the second paper there is a discussion of the lateral diffusion of a circular vortex in a medium composed of a single layer. The analysis depends on two integrals of type

$$\int_0^\infty \exp[-(\sigma^2 + \frac{1}{4}\sigma^4 s_n)/4\beta^2] J_0(\sigma s) \sigma ds, \quad n=1, 2,$$

and is illustrated by means of graphs. In an appendix a solution of

$$\frac{\partial H}{\partial s} + \frac{\partial^2}{\partial \eta^2} \left(\eta^2 \frac{\partial^2 H}{\partial \eta^2} \right) = 0$$

is derived.

H. Bateman (Pasadena, Calif.).

Blinova, E. N. A hydrodynamical theory of pressure and temperature waves and of centres of atmosphere action. C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 257-260 (1943). [MF 9830]

It is assumed that an undisturbed zonal current of constant angular velocity α exists in an incompressible atmosphere. Pressure \bar{p} and temperature \bar{T} vary with the latitude. Superimposed on this current is a small perturbation without a vertical velocity component. The temperature of an individual parcel of air is assumed to remain constant so that (t time, λ longitude, θ colatitude, R radius of the earth, letters without bar perturbation quantities)

$$\left(\frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial \lambda} \right) T = \frac{1}{R^2 \sin \theta} \frac{\partial T}{\partial \theta} \frac{\partial \psi}{\partial \lambda};$$

ψ is a streamfunction whose existence is assumed. By cross differentiation of the two horizontal equations of motion (ω angular velocity of the earth's rotation)

$$\left(\frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial \lambda} \right) \left[\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 \psi}{\partial \lambda^2} \right] + 2(\alpha + \omega) \sin \theta \frac{\partial \psi}{\partial \lambda} = \frac{1}{\rho \bar{T}} \left[\frac{\partial \bar{p}}{\partial \theta} \frac{\partial T}{\partial \lambda} - \frac{\partial T}{\partial \theta} \frac{\partial \bar{p}}{\partial \lambda} \right];$$

\bar{p} and \bar{T} are mean value of temperature and density for the whole atmosphere. The second term on the right-hand side is neglected since it is found to be considerably smaller than the first term. The perturbation quantities ψ, p and T are regarded as consisting of two terms each, the first ones ψ', p', T' being independent of time and representing monthly means while ψ'', p'' and T'' stand for the non-steady components. Then T' is regarded as known and expressed by a series of surface harmonics with coefficients δ_n^m . By means of the preceding equation, ψ' is then obtained as a series of surface harmonics whose coefficients are

$$C_n^m = \frac{2R^2\omega((n-m)/(2n-1))\delta_{n-1}^m + ((m+n+1)/(2n+3))\delta_{n+1}^m}{\bar{T} \cdot 2((\alpha+\omega)/\alpha) - n(n+1)}.$$

For suitable values of α and n a special form of resonance may occur. Moreover, p' can be obtained from one of the equations of motion. The nonsteady components ψ'', T'' and p'' are found by first determining p'' at the initial time so that $\bar{p} + p' + p''$, when $t=0$, corresponds to the actual pressure distribution. The other quantities ψ'' and T'' can then be determined from the series for p'' . Computation of the monthly mean distribution of p' at sea level and at 4000m elevation from the monthly mean distribution of T' gave good agreement with the actually observed pressure distribution. B. Haurwitz (Cambridge, Mass.).

Esclangon, Ernest. Sur la réflexion et la réfraction d'ondes acoustiques à la surface de séparation de deux fluides en repos ou en mouvement relatif de translation. C. R. Acad. Sci. Paris 215, 45-48 (1942). [MF 9476]

The scripts $i=1, 2$ refer to quantities in two media separated by $z=0$. Let a_i be sound velocity and let $\vec{v}_i=0$, $\vec{v}_2=(\alpha, \beta, 0)$ be the medium velocities. Let I, R, ϕ be the velocity potentials for incident, reflected and refracted plane waves when the angle of incidence is θ_i . The following conditions are used to determine the resulting phenomena: (1) ϕ is invariant under $x_1 \sin \theta_1 + at = x_1' \sin \theta_1 + at'$; (2) $x_2 = x_1 + at$, $y_2 = y_1 + at$; (3) $\nabla^2 \phi = (1/a^2) \phi_{tt}$; (4) the normal velocity on both sides of $z=0$ is the same; (5) the pressure on both sides of $z=0$ is the same. If there is a true refracted wave, it is plane and

$$(6) \quad \phi = \phi(x_2 \sin \theta_1 + (\omega_1 + \alpha \sin \theta_1)t + \lambda z).$$

The relations (1) ... (6) determine completely all the salient features. For instance,

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{c_1}{a_2} \left(1 + \frac{\alpha}{a_1} \sin \theta_1 \right).$$

It turns out that R may be 0. On the other hand, if λ is imaginary, there is total reflection. For this case the author resorts to a skillful application of functions of a complex variable to determine the quantities of interest. (As might have been expected, the disturbance in medium (2) is now exponentially damped with z .) D. G. Bourgin.

Cremer, L. Theorie der Schalldämmung dünner Wände bei schrägem Einfall. Akustische Z. 7, 81-104 (1942). [MF 9410]

Much of the paper repeats essentially well-known results. The author contributes the suggestion for applied acoustics that a wall be considered either (1) a membrane or (2) a plate. Crude approximations are used in developing the consequences which seem of some practical interest. Write (3) $p_1 - p_2 + X(w) = mw_{tt}$, where $X = S \partial^2/\partial x^2$ or $-B \partial^4/\partial x^4$

for case (1) or (2), respectively, w is the wall displacement, p_1 and p_2 are pressures on opposite sides of the wall and m , S , B are measured per unit area since the thickness is supposed constant and the wall is of infinite extent. Write c , c_1 for air and wall sound velocities. Assume a weak incident wave at angle θ and circle frequency ω and use bars for the time free factors. Then the induced velocity amplitude in the wall is (4) $\bar{v} \sim e^{k_1 i(\omega/c) \sin \theta x}$. Since $\bar{v} = i\omega \bar{w}$, we have, from (3),

$$(5) \quad \bar{p}_1 - \bar{p}_2 = [i\omega(n - X((\sin \theta)/c))] \bar{v}.$$

The parameter $(\bar{p}_1 - \bar{p}_2)/\bar{v}$ is called the transmission resistance and is used generally in acoustics. For (1), $c_1 = (S/m)^{1/2}$. For (2), $c_1 = (B\omega^2/m)^{1/2}$. Hence

$$(6) \quad T = i\omega m(1 - ((c_1/c) \sin \theta)^{2q}),$$

$q=1$ for (1) and $q=2$ for (2). This is the key result. In particular, $T=0$ for $c_1=c/\sin \theta$ and the name "coincidence" is coined for such a happy situation. Physical applications are discussed.

D. G. Bourgin (Urbana, Ill.).

Backhaus, H. Zur Berechnung des Schallfeldes der kreisförmigen Kolbenmembran. Z. Tech. Phys. 24, 75-78 (1943). [MF 9735]

In an earlier paper [Ann. Physik (5) 5, 1-35 (1930)] the author obtained a formula for the velocity potential of a circular piston diaphragm in the form of a series of Legendre polynomials of even order. Later, L. V. King [Canadian J. Research 11, 135-155 (1934)] gave an alternative solution derived from the wave equation for the potential. The author now shows that the two solutions are not inconsistent, as King implied, and that his original solution can be obtained more simply by King's method.

M. C. Gray (New York, N. Y.).

Theory of Elasticity

Luntz, I. L. Sur la flexion des plaques longues encastrees. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 167-178 (1943). (Russian. French summary) [MF 10074]

The problem consists in a solution of the equation (*) $(\text{div grad})^2 w(x, y) = f(x, y)$, where w is the deflection of the plate and $f(x, y)$ is proportional to the intensity of the load which is assumed to act normally to the plate. The author represents the boundary in the form $y=cY(x)$, where the constant c is a measure of the width of the plate. He changes the coordinates (x, y) into (x, Y) and puts $f = \sum c^i Y^i f_i(x)$, $(\dagger) w = \sum c^i w_i(x, Y)$. The substitution of this into (*) gives a recurrent system of differential equations for the w_i 's. The boundary conditions are taken for each w_i the same as for w , that is, $w = \partial w / \partial x = \partial w / \partial y = 0$. The author gives explicit expressions for w_1, \dots, w_3 and a general recurrent relation for the w_i 's in terms of the f_i 's. He states that the method is in general divergent, but he shows that it may be used for sufficiently narrow plates. He gives an estimate of the difference between the exact value of w and the n th partial sum of (\dagger) . Such estimate is calculated by representing w by means of the Green function G of the problem; by showing that, if G' is the Green function of a similar problem for an infinite plate which contains the given one in its interior, then $|G| < |G'|$ and finally by using Nadai's result [Bauingenieur 2, 299-304 (1921)] to evaluate G' . As an example the author calculates the deflec-

tion of a plate bounded by two parabolas symmetric with respect to the line joining their points of intersection, the load being uniformly distributed, and shows that the first two nonzero terms of (\dagger) solve the problem with an error less than 8 per cent if the ratio of the width of the plate to its length is less than 1/9. In the case of an elliptic plate under uniform load the method gives exactly a known formula.

I. Opalowski (Chicago, Ill.).

Chien, Wei-Zang. The intrinsic theory of thin shells and plates. I. General theory. Quart. Appl. Math. 1, 297-327 (1944). [MF 9908]

This paper contains a systematic treatment of the general problem of the thin shell, which includes the problem of the thin plate as a special case. The material of the shell is assumed to be elastically isotropic and homogeneous. Stress, strain and change of curvature appear explicitly in the treatment, while the displacement does not. The tensor notation is employed.

In a previous paper [J. L. Synge and W. Z. Chien, Theodore von Kármán Anniversary Volume, California Institute of Technology, Pasadena, Calif., 1941, pp. 103-120; these Rev. 3, 30] the foundations of the work were laid. There were introduced two macroscopic stress tensors (defined by integration of the stresses over the thickness of the shell), the macroscopic equations of equilibrium (the equilibrium conditions in terms of the macroscopic stress tensors), the microscopic (usual) equations of equilibrium and the equations of compatibility. In the present paper the macroscopic equations of equilibrium and three of the equations of compatibility are transformed into six equations for the six unknowns $p_{\alpha\beta}, q_{\alpha\beta}$ ($\alpha, \beta=1, 2$), which represent the extension and change of curvature of the middle surface. When these quantities have been found, the strain and stress can be determined from the three microscopic equations of equilibrium and the three remaining equations of compatibility.

No assumptions are made regarding the thickness of the shell or the magnitude of the displacement. Expansions in the thickness of the shell are employed, but the remainder terms are carefully recorded. In parts II and III [to appear presently in the Quarterly of Applied Mathematics], it is proposed to discuss the various approximate forms of the equations arising from consideration of the thinness of the shell and the smallness (or vanishing) of its curvature.

G. E. Hay (Providence, R. I.).

Deuker, Ernst-August. Zur Stabilität der elastischen Schalen. I. Z. Angew. Math. Mech. 23, 81-100 (1943). [MF 9863]

The author applies the theory developed in an earlier paper [Deutsche Math. 5, 546-562 (1941); these Rev. 3, 95] to the problem of the stability of thin elastic shells. The theory is developed on the basis of the two principal assumptions commonly made in dealing with shells: (1) normals to the undeformed middle surface become normals to the deformed middle surface; (2) the normal stress on elements parallel to the middle surface is so small that its effects on strains can be neglected. The theory (which is essentially a nonlinear one) is developed in a very general and systematic way, with the use of the greater part of the fundamental formal apparatus of differential geometry, expressed in the tensor notation. It might be noted specifically that quadratic terms in the expressions for the strains in terms of the displacements are not neglected. However,

because of the two assumptions noted above, the treatment is far less general than that given in Chien's paper reviewed above. As a specific example to illustrate the theory the author treats the problem of the stability of the cylindrical shell. To do so, the author makes a number of assumptions in addition to those noted above which lead to a linearization of the problem. The well-known results for the buckling of the cylindrical shell are then obtained. *J. J. Stoker.*

Moushtary, C. M. The approximate solution of certain problems of stability of a thin-walled conic shell with a circular cross-section. *Appl. Math. Mech.* [Akad. Nauk SSSR. Prikl. Mat. Mech.] 7, 155-166 (1943). (Russian. English summary) [MF 10073]

The author makes use of the Ritz method to obtain an approximate solution of several problems of stability of a thin conical shell, of circular cross-section, subjected to uniform compressive and shearing stresses along the butt ends. It is assumed that $u=v=w=0$ for $r=r_0$ and $r=r_0+l$, where u is the displacement along the generator of the cone, v is the displacement along the tangent to the section $r=\text{const.}$, w is the displacement along the normal to the middle surface, l is the length of the shell and r is the distance from the vertex of the cone to a point on the middle surface. The condition of clamping $w_r=0$ is disregarded. The author specializes his formulas for critical stresses to several cases of cylindrical shells and plates and finds his results in substantial agreement with those of earlier investigators. *I. S. Sokolnikoff* (Madison, Wis.).

Ghosh, S. Stress systems in rotating aeolotropic discs. *Bull. Calcutta Math. Soc.* 35, 61-65 (1943). [MF 9950]

The paper deals with the stresses in a rotating aeolotropic disk with circular or elliptic boundary. The material is supposed to possess three orthogonal planes of symmetry at each point, one of these planes being parallel to the plane faces of the disk. If the rim of the disk is free from stresses, a state of plane stress is seen to exist in the disk.

W. Prager (Providence, R. I.).

Templeton, Haydn. Approximate solution for tapered pin-ended struts. *J. Royal Aeronaut. Soc.* 48, 6-11 (1944). [MF 9917]

Approximate solutions for the critical load of an axially loaded strut and for the deflection produced in an eccentrically loaded strut are obtained by the energy method. For the axially loaded strut, the deflection $y=a \sin (\pi x/l)$ is assumed. The strain energy of the strut is computed and equated to the work done by the load to give a value for the critical load. For the eccentrically loaded strut, the deflection $y=cx(l-x)+a \sin (\pi x/l)$ is assumed. By proper choice of c , this can be made to fit the end conditions; a is changed by δa and the corresponding work done by the load is equated to the corresponding increase in the strain energy of the strut to give a value for a ; the deflection y is thus determined. For most practical purposes, these solutions compare favorably with solutions based on successive approximation, since for the same accuracy the time required for a solution by successive approximation is much greater. *G. E. Hay* (Providence, R. I.).

Greenspan, Martin. Axial rigidity of perforated structural members. *J. Research Nat. Bur. Standards* 31, 305-322 (1943). [MF 9698]

The change of length ϵ of a uniform rod under tension or compression is given by the formula $\epsilon=PL/R$, in which P

is the total axial load, L the length of the rod and $R=EA$ (E modulus of elasticity and A cross section area of the rod) is a quantity called by the author the axial rigidity of the rod. The object of the paper is to generalize the formula for the change of length to rods of narrow rectangular cross section in which a series of uniformly spaced holes of circular, elliptical or other shapes have been cut. Formulas for the rigidity R in such cases are obtained using the theory of plane stress. This is made practicable by assuming that the diameter of the holes is small compared with the breadth of the rod and the spacing of the holes. The theory has been found to agree very well with test results.

J. J. Stoker (New York, N. Y.).

Denke, Paul H. Strain energy analysis of incomplete tension field web-stiffener combinations. *J. Aeronaut. Sci.* 11, 25-40 (1943). [MF 9781]

This paper deals with a thin web beam of uniform depth, stiffened by vertical stiffeners and parallel caps along the upper and lower edges. The beam is subjected to shear loads such that each panel is deformed into an approximate parallelogram. It is assumed that the vertical stiffeners remain straight and are compressed, the caps are bent and compressed and the web forms a series of diagonal wrinkles. The potential energy of the deformed beam is computed in terms of seven parameters specifying the deformation. Then, by the principle of minimum potential energy, the equilibrium configuration is deduced. The results of this analysis are compared in part with experimental values and are found to possess sufficient accuracy for design purposes.

G. E. Hay (Providence, R. I.).

Hildebrand, F. B. On the stress distribution in cantilever beams. *J. Math. Phys. Mass. Inst. Tech.* 22, 188-203 (1943). [MF 9777]

This paper deals with the determination of the stress in a flanged or unflanged cantilever beam with a narrow rectangular cross-section, the material being assumed rigid in the direction of the long axis of the cross-section. The loading is in the plane of the web plate and a state of plane stress is assumed. The elementary theory of this problem yields a simple solution which, however, does not satisfy exactly the boundary conditions at the end of the beam. In this paper these boundary conditions are taken into account. First, the method of least work is employed to obtain a stress function $\phi(x, y)$ in the form $\phi = F_s + F$, where F_s is the stress function corresponding to the elementary theory. This solution is only approximate, since it fails to satisfy the compatibility condition and one boundary condition. Secondly, an exact solution is obtained, in which F is determined in the form $F = \sum X_n(x) Y_n(y)$.

When the external loading consists of a concentrated load on the free end, it is found that: (1) in both cases σ_s/σ_0 is very nearly a linear function of h/l , where σ_s is the maximum stress, σ_0 is the elementary maximum stress, $2h$ is the web depth and l is the length of the beam; (2) the two linear relations differ by little, which gives an indication of the validity of the approximate method of least work; (3) for $h/l \ll 1$, $\sigma_s/\sigma_0 \approx 1$. *G. E. Hay* (Providence, R. I.).

Hoff, N. J. A strain energy derivation of the torsional-flexural buckling loads of straight columns of thin-walled open sections. *Quart. Appl. Math.* 1, 341-345 (1944). [MF 9911]

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